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Optimally defined Racah-Casimir operators for $su(n)$ and their eigenvalues for various classes of representations

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Abstract

This paper deals with the striking fact that there is an essentially canonical path from the i-th Lie algebra cohomology cocycle, $i = 1, 2, \ldots l$, of a simple compact Lie algebra $\mathfrak{g}$ of rank l to the definition of its primitive Casimir operators $C^{(i)}$ of order m_i. Thus one obtains a complete set of Racah-Casimir operators $C^{(i)}$ for each $\mathfrak{g}$ and nothing else. The paper then goes on to develop a general formula for the eigenvalue $c^{(i)}$ of each $C^{(i)}$ valid for any representation of $\mathfrak{g}$, and thereby to relate $c^{(i)}$ to a suitably defined generalised Dynkin index. The form of the formula for $c^{(i)}$ for $su(n)$ is known sufficiently explicitly to make clear some interesting and important features. For the purposes of illustration, detailed results are displayed for some classes of representation of $su(n)$, including all the fundamental ones and the adjoint representation.

1 Introduction

It is well-known (see *e.g.* [1, 2, 3] for lists of references and further details) that the l basis elements of the Lie algebra cohomology of a simple compact Lie algebra $\mathfrak{g}$ define, up to a constant, l totally antisymmetric tensors. In fact, these may also be understood as the coordinates of the different invariant $(2m-1)$-forms on the manifold of the compact group G associated with $\mathfrak{g}$ that, in the Chevalley-Eilenberg version of the Lie algebra cohomology [4], characterise the $(2m-1)$-cocycles. Given a simple compact $\mathfrak{g}$, we shall refer to these l tensors as the Omega tensors $\Omega^{(2m_s-1)}$ of $\mathfrak{g}$. They have orders $2m_s - 1$, $s = 1, 2, \ldots, l$, where m_s are the orders of the primitive Casimir-Racah operators of $\mathfrak{g}$ (see also [1, 2, 3] for lists of references). For $su(n)$, $m_s \in \{2, 3, \ldots, n\}$, and hence the Omega tensors are of orders $3, 5, \ldots (2l+1)$. There is an essentially canonical path from the Omega tensors of a given $\mathfrak{g}$ that leads to the set of its l *primitive* Racah-Casimir operators $C^{(m_s)}$. Following this path [2], the resulting set of Racah-Casimir operators $C^{(m_s)}$ (represented by invariant symmetric tensors $t^{(m_s)}$ of order m_s) is optimally defined in the sense that it contains one member for each required order m_s and nothing else. The procedure allows for the appearance of no $C^{(m_s)}$ other the l primitive ones: any formal attempt to define $C^{(m)}$ for, say, $su(n)$ for $m > n$ simply produces a vanishing result. Since this paper concentrates on $su(n)$, we shall not worry about the refinements that are needed to deal explicitly with all the invariants of the even orthogonal algebras $\mathfrak{g} = D_l$, where the Pfaffian enters the picture. Nor will the subsequent discussion make explicit the qualifications that may be needed to cover the exceptional algebras.

The paper proceeds from the definition of a complete set of primitive Racah-Casimir operators for $\mathfrak{g}$ to a new general result for the eigenvalues $c^{(m_s)}(D)$ of $C^{(m_s)}(D)$ for a generic representation D

$$X_i \mapsto D(X_i) \tag{1}$$

of the Lie algebra

$$[X_i\,,\,X_j] = i f_{ijk} X_k \tag{2}$$

of $\mathfrak{g}$. We have here written f_{ijk} for the structure constants of $\mathfrak{g} = su(n)$. For this algebra, almost all of the technical machinery is at hand [2, 5] to enable us to display explicitly the key features of our general result for $c^{(m)}(D)$. Obvious analogues of these results are applicable to all other $\mathfrak{g}$.

Our main result states that, for any representation D,

$$(\dim D)\, c^{(m_s)}(D) = 2^{1-m_s} (gdi)^{(m_s)}(D)\, \Omega^{(2m_s-1)^2} \tag{3}$$

where

$$\Omega^{(2m_s-1)^2} \equiv \Omega_{i_1\ldots i_{2m_s-1}} \Omega_{i_1\ldots i_{2m_s-1}} \quad , \tag{4}$$

and $(gdi)^{(m_s)}(D)$ is a number dependent on the order m_s of the Racah-Casimir, the representation D considered and $\mathfrak{g}$, or rather in the case of $su(n)$, on n. For the representation considered, $(gdi)^{(m_s)}(D)$, $s = 1, \ldots, l$, is an acronym for s-th *generalised Dynkin index* for the representation D considered and its use in (3) is discussed below. What is special about $su(n)$ is that $\Omega^{(2m_s-1)^2}$ is known explicitly for all n and for all $2 \leq m \leq n$ (from now on, we drop the subindex $s = 1, \ldots, l$ in m_s). From [5] we quote

$$\Omega^{(2m-1)^2} = \frac{2^{2m-3}}{(2m-2)!}\, n \prod_{r=1}^{m-1} (n^2 - r^2) \tag{5}$$

$$= \frac{4}{(2m-2)(2m-3)}\, \Omega^{(2m-3)^2} \quad . \tag{6}$$

Eq. (5) exhibits features of (3) which we believe apply equally well to all other $\mathfrak{g}$. Eq. (5) shows that $\Omega^{(2m-1)^2} \neq 0$ and hence $\Omega^{(2m-1)}$ is non-vanishing only of $m \leq n$. In other words the primitive $(2m-1)$-cocycle exists only for $m \leq n$ as known from Lie algebra cohomology, and (3) gives a null result for $c^{(m)}(D)$ only when $n < m$. The power of two in (3) has been chosen, as far as we know it to be necessary, to ensure that, as is customary for an index, $(gdi)^{(m)}(D)$ takes on only integral values. In the case of $m = 2$ and the familiar Dynkin index itself [6], (3) takes on its standard form (see *e.g.* [7])

$$(gdi)^{(2)}(D) = \frac{2 \dim D}{n \dim \mathfrak{g}}\, c^{(2)}(D) \quad , \tag{7}$$

using $f_{ijk}f_{ijk} = \Omega^{(3)^2} = n(n^2-1) = n.\dim(su(n))$. The factor n in the denominator of (7) reflects the fact that for uniformity (in m) of our definitions of the various $C^{(m)}$ for $su(n)$, we have defined the quadratic Casimir operator of $su(n)$ as

$$C^{(2)} = nX_iX_i \quad , \quad i = 1, \ldots, n-1 \quad , \tag{8}$$

see Sec. 2.3. For higher values of m there is less agreement as to how the Casimir operators $C^{(m)}$, and hence the $(gdi)^{(m)}(D)$, should be defined. We have argued that our definition of the former is optimal, featuring as it does t-tensors [2] and Omega tensors that are in one-to-one correspondence with the cohomology cocycles of $su(n)$. This is tantamount to asserting that the Omega tensors are the fundamental entities (in fact, Ω^3 is given by the structure constants of the algebra themselves), and our definition of Casimir operators follows from this and reflects it too. A good recent account of the role of, and of one way of defining, generalised Dynkin indices, which were originally introduced in [8], is [9], which refers to earlier papers [10] [11]. The paper [9] also emphasises the role of 'orthogonal' tensors –in essence our t-tensors– making reference to [2] in this context, and attributing the recognition of the importance of orthogonality to the definition of generalised Dynkin indices to [8]. Ref. [9] contains extensive tabulations of generalised Dynkin indices, as does [12]. Another useful discussion of indices is contained in [16], which aims, as we do here, at getting results for all $su(n)$ valid for all n, using procedures –Cvitanovic's bird-track methods– that are there also applied to other $\mathfrak{g}$. Our work differs from that of the papers just cited in that it emphasises the central role of the Omega-tensors, and employs a definition of indices that follows from this viewpoint. In view of (5) we believe that a significant amount of new information is contained in our work. We would further wish to advocate that (3) –the formula for the eigenvualue $c^{(m)}(D)$ of the Racah-Casimir operator of oder m– be seen as the result of primary importance. The $(gdi)^{(m)}(D)$ are merely numbers, knowledge of which is required to complete the determination of the $c^{(m)}(D)$. Thus while the number $(gdi)^{(m)}(D)$ in some sense characterises the eigenvalue $c^{(m)}(D)$, general $su(n)$ formulas for $(gdi)^{(m)}(D)$ do not automatically exhibit the restrictions on n necessary for their applicability. Thus, for the adjoint represention ad, see below, of $su(n)$,

$$(gdi)^{(4)}(ad) = 2n \tag{9}$$

But this applies *only* when $n \geq 4$, since (3)–(5) show that $c^{(4)}(D)$ equals zero for any D for $n = 2, 3$, as it should do, since $\mathcal{C}^{(4)}$ is absent for these n-values. See also comments following (70) and (103) below.

The paper turns next to providing some illustrations of the results that are contained within (3). We wish to deduce the values of $(gdi)^{(m)}(D)$ for various m and representations D, obtaining results from a single computation that are valid for all n. For this purpose we consider the following classes of representations of $su(n)$. We will use the notation $\mathcal{F}^s$ for $s = 1, 2 \ldots, l$, for the fundamental representations of $su(n)$ of rank $l = n-1$ (writing also $\mathcal{F} \equiv \mathcal{F}^1$ for the defining representation), and denote the adjoint representation of the algebra by ad.

• *The defining representation of* $su(n)$, $\mathcal{F}$

$\mathcal{F} = (1, 0, \ldots, 0)$ and is given by

$$X_i \mapsto \tfrac{1}{2}\lambda_i \equiv \mathcal{F}_i \quad , \tag{10}$$

where the λ_i are the standard Gell-Mann matrices of $su(n)$ [13]. Using results from [5], we can show by a single calculation that

$$(gdi)^{(m)}(\mathcal{F}) = 1 \tag{11}$$

for all n and for all $m \leq n$. This simple general result offers some indication of the appropriateness of our definition of $(gdi)^{(m)}$.

• The *adjoint representation ad of* $su(n)$

In Dynkin coordinates $ad = (1, 0, \ldots, 1)$ (see *e.g.* [7]) and it is given by

$$X_i \mapsto ad(X_i) \equiv ad_i \quad , \quad (ad_i)_{jk} = -i f_{ijk} \quad . \tag{12}$$

We show that $(gdi)^{(m)}(\mathcal{F}) = 0$ for all odd m for all n, and give results for $m = 2, 4$ and 6, on the basis of one calculation for each of these three m values.

• *The representation* $\mathcal{D}$ *of* $\mathfrak{g} = su(n)$

Let $\mathcal{D}$ denote the representation of $\mathfrak{g}$ built [14] [15] using the set of hermitian Dirac matrices of a euclidean space of dimension $\dim \mathfrak{g} = r$. The representation $\mathcal{D}$ is defined by means of

$$X_i \mapsto \mathcal{D}(X_i) \equiv S_i = -\tfrac{1}{4} i f_{ijk} \gamma_j \gamma_k \quad , \tag{13}$$

in terms of Dirac matrices such that

$$\{\gamma_i \, , \, \gamma_j\} = 2\delta_{ij} \quad ; \tag{14}$$

hence, $\mathcal{D}$ has dimension $2^{[r/2]}$ ($[x]$ denotes the integral part of x). Unlike all the other representations that we treat $\mathcal{D}$ is reducible; it describes the direct sum of a number of copies of the irreducible representation of $\mathfrak{g}$ whose highest weight is the principal Weyl vector $\delta = \frac{1}{2}\sum_{\alpha>0} \alpha$, *i.e.* half the sum of the positive roots of $su(n)$, given by $\delta = (1, \overset{l}{\ldots}, 1)$ in Dynkin coordinates. This representation has dimension $2^{(r-l)/2}$. The actual number of copies of δ in $\mathcal{D}$ is $2^{[l/2]}$. It follows that $\dim \mathcal{D} = 2^{[l/2]} \, 2^{(r-l)/2} = 2^{[r/2]}$. We restrict to $\mathfrak{g} = su(n)$, for which $l = n-1$ and $r = n^2 - 1$. We show that $(gdi)^{(m)}(\mathcal{D})$ is zero for all odd m for all $su(n)$, and give explicit results for $m = 2$ and $m = 4$ only.

• *The* l *fundamental representations* $\mathcal{F}^s$ *and the irrep* $\mathcal{S}_p = (p, 0, \ldots, 0)$.

By dealing with the completely symmetrised and also the completely antisymmetrised direct products of p copies of the defining representation $\mathcal{F}$, we derive results for the representations of $su(n)$ with Dynkin coordinates $(p, 0, \ldots, 0)$, and for the fundamental representations $\mathcal{F}^s, \quad s = 1, 2, \ldots, n-1$. For the latter, we give all indices for all $\mathcal{F}^s$ for $n = 3, 4, 5$ and 6. Bird-track methods [16] are employed here.

The material of this paper is organised as follows. Sec. 2 two gives a brief description of the various families of $su(n)$ tensors, including the Omega tensors, that are involved in the build up to the definition of Casimir operators. To some extent this reviews our earlier paper [2], where detailed references to previous studies of Casimir operators may be found. In Sec. 3, we derive and discuss the key result (3). In Sec. 4, we turn to the illustration of (3) for the classes of $su(n)$ representations just listed.

The interest of this paper in the eigenvalues of Casimir operators has been in the context of generalised Dynkin indices, because our approach brings these in completely naturally. There are however other sources of information on the subject. There is [17] where valuable explicit formulas are given for all Casimir operators of all classical algebras and also for g_2, while [18] addresses the problem for other exceptional algebras.

2 Definitions of tensors and Racah-Casimir operators

2.1 The d-tensors

This is a family of symmetric tensors first defined by Sudbery [19]. The definition sets out from the standard Gell-Mann totally symmetric third order tensor d_{ijk} that exists for all $n \geq 3$ and is traceless $d_{ijk}\delta_{ij} = 0$. Higher order tensors in the family

$$d^{(r)}_{(i_1,\ldots,i_r)} \quad , \tag{15}$$

are defined recursively by symmetrising

$$d^{(r)}_{i_1,\ldots,i_r} = d^{(r-1)}_{i_1,\ldots,i_{r-2}j} d^{(3)}_{j i_{r-1} i_r} \quad , \tag{16}$$

over all its $i_1, \ldots, i_r$ indices. Round brackets here denote symmetrisation with unit weight. Thus

$$\begin{aligned} d^{(3)}_{(ijk)} &= d_{ijk} \\ d^{(4)}_{(ijkl)} &= \tfrac{1}{3}(d_{ijp}d_{pkl} + d_{jkp}d_{pil} + d_{kip}d_{pjl}) \quad . \end{aligned} \tag{17}$$

Sometimes it is useful to refer to $d^{(2)}_{ij} = \delta_{ij}$ as the rank two member of the family.

2.2 The Omega tensors

Using the mentioned correspondence between $(2m-1)$-cocycles and $\Omega^{(2m-1)}$ tensors, we have (see, e.g. [1, 2] for the structure of these expressions)

$$\Omega^{(3)}_{ijk} = f_{ijk} = f_{aij} d^{(2)}_{(ak)} \quad , \tag{18}$$

$$\Omega^{(5)}_{ijkpq} = f_{a[ij} f^{b}{}_{kp]} d^{(3)}_{(abq)} \quad , \tag{19}$$

$$\Omega^{(7)}_{ijkpqrs} = f_{a[ij} f^{b}{}_{kp} f^{c}{}_{qr]} d^{(4)}_{(abcs)} \quad , \tag{20}$$

and in general

$$\Omega^{(2m-1)}_{i_1 j_i \ldots i_{m-1} j_{m-1} k} = f^{k_1}{}_{[i_1 j_1} \ldots f^{k_{m-1}}{}_{i_{m-1} j_{m-1}]} d^{(m)}_{(k_1 \ldots k_m)} \quad . \tag{21}$$

Here square brackets indicate unit weight antisymmetrisation over all the surrounded indices. The structure of $\Omega^{(2m-1)}$ above is general for any $\mathfrak{g}$; what makes it specific to $su(n)$ are the orders $(3, 5, \ldots, (2l+1))$, and of course the fact that the f's are the $su(n)$ ones. One may check explicitly that $\Omega^{(2m-1)}$ tensor is fully antisymmetric in all its $(2m-1)$ indices $(i_1 j_1 i_2 j_2 \ldots i_{m-1} j_{m-1} k)$, even if only the first $(2m-2)$ indices are antisymmetrised by actual square brackets. The position of the indices in this paper is without metric signifance. Since $\mathfrak{g}$ is compact and its generators are hermitian, we may take the Killing metric as the unit matrix, and so the raising of indices may just serve to remove them from the sets of indices that are subject to antisymmetrisation (or symmetrisation).

A detailed account of the properties of Omega tensors has recently been prepared [5]. The extensive compilation of results contained in [5] includes the important formula (5), together with its derivation. As noted above, (5) makes clear that $\Omega^{(2m-1)}$ is absent for $su(n)$ whenever $m > n$.

2.3 The t-tensors

Following [2] we review some properties of this family of totally symmetric and totally tracelsss tensors $t^{(m)}$ for $su(n)$. The definitions are

$$t^{(2)}_{ak} = \Omega^{(3)}_{ijk} f_{ija} \quad , \tag{22}$$

$$t^{(3)}_{abm} = \Omega^{(5)}_{ijklm} f_{ija} f_{klb} \quad , \tag{23}$$

$$t^{(4)}_{abcq} = \Omega^{(7)}_{ijklmpq} f_{ija} f_{klb} f_{mpc} \quad , \tag{24}$$

$$t^{(5)}_{abcds} = \Omega^{(9)}_{ijklmpqrs} f_{ija} f_{klb} f_{mpc} f_{qrd} \quad , \tag{25}$$

and in general

$$t^{(m)}_{k_1 \ldots k_m} = \Omega^{(2m-1)}_{i_1 j_i \ldots i_{m-1} j_{m-1} k} f_{i_1 j_1 k_1} \ldots f_{i_{m-1} j_{m-1} k_{m-1}} \quad . \tag{26}$$

It follows from definition (26) that the tensor $t^{(m)}$ is fully symmetric ([2], lemma 3.2)

We do not make extensive use of explicit expressions for the t-tensors, but it is useful to note results from [2, 5]

$$t^{(2)}_{ij} = n\delta_{ij} \quad , \tag{27}$$

$$t^{(3)}_{ijk} = \frac{1}{3} n^2 d_{ijk} \quad , \tag{28}$$

$$t^{(4)}_{ijkl} = \frac{1}{15}(n(n^2+1)d^{(4)}_{(ijkl)} - 2(n^2-4)\delta_{(ij}\delta_{kl)}) \quad , \tag{29}$$

$$t^{(5)}_{ijklm} = \frac{n}{135}((n(n^2+5)d^{(5)}_{(ijklm)} - 2(3n^2-20)d_{(ijk}\delta_{lm)}) \quad . \tag{30}$$

We have adjusted the normalisations in (27)–(30) by excluding some powers of two present in (6.12)–(6.14) of [2]. The t-tensors are 'orthogonal' among themselves, which means that the maximal contraction of a $t^{(m)}$ with a tensor $t^{(m')}$ of different order yields zero. This implies, in particular, that $t^{(m)}$ is traceless with respect any two indices. In the simplest, third order case, eq. (28), this is just the well known property $d_{ikk} = 0$. For order four, $t^{(4)}$, this means that

$$t^{(4)}_{ijkl}\delta_{ij} = 0 \quad , \quad t^{(4)}_{ijkl} d_{ijk} = 0 \quad . \tag{31}$$

But, since the trace formulas for d-tensors give

$$d^{(4)}_{(ijkl)} d_{ijm} = \tfrac{2}{3}\tfrac{(n^2-8)}{n} d_{klm} \quad , \tag{32}$$

we learn that the non-trivial result for the non-maximal contraction

$$t^{(4)}_{ijkl} d_{ijm} = \tfrac{2}{45} n^2(n^2-9) d_{klm} \quad , \tag{33}$$

holds, although it collapses to zero, as it ought, for $n = 3$. Also (33) yields the second part of (31) when one makes the contraction $k = m$. For order five, the orthogonality of the t-tensors means that

$$t^{(5)}_{ijklm} t^{(2)}{}_{ij} = 0 \quad , \quad t^{(5)}_{ijklm} t^{(3)}_{ijk} = 0 \quad , \quad t^{(5)}_{ijklm} t^{(4)}{}_{ijkl} = 0 \quad , \tag{34}$$

and so on.

One way to see that the t-tensors, like the Omega tensors, are absent for $m > n$ is to consider the fully contracted square of a generic t-tensor

$$t^{(m)^2} = t^{(m)}_{k_1 \ldots k_m} t^{(m)}_{k_1 \ldots k_m} \quad . \tag{35}$$

This scalar quantity can be seen to contain the same product of factors as is seen in (5). Thus it is a polynomial in n that has factors which vanish whenever $n < m$. Actually the proof of this claim for the t-tensors was achieved for $m \leq 5$ in [2], whereas (5) is proved in full generality in [5], relatively speaking rather easily.

The d-tensors are less convenient than the t-tensors in that for $su(n)$ they are well-defined for any order m, but are present as non-primitive tensors for $m > n$. However [19] [2] [20] the unwanted or rather inessential d-tensors of higher orders can always be expressed in terms of the primitive set with $m \leq n$. For example for $su(3)$ we have

$$d^{(4)}_{(ijkl)} = \tfrac{1}{3}\delta_{(ij}\delta_{kl)} \quad , \tag{36}$$

$$d^{(5)}_{(ijklm)} = \tfrac{1}{3}\delta_{(ij} d_{klm)} \tag{37}$$

and, for $su(4)$,

$$d^{(5)}_{(ijklm)} = \frac{2}{3}\delta_{(ij}d_{klm)} \quad . \tag{38}$$

The above expressions exhibit the non-primitive character of the symmetric tensors on their left hand sides. It becomes increasingly hard to supply such results for $su(n)$ at higher n. Fortunately this is unnecessary. Further, as [5] shows, the d-tensors serve perfectly well for the definition of Omega tensors. If non-primitive d-tensors like (36)–(38) are employed in the definition of tensors like those of Sec.2.2, when we know that no Omega tensor is allowed ([2], Cor. 3.1), a vanishing result is obtained [5]. In follows that symmetric invariant tensors differing in non-primitive parts lead to the same Omega tensors.

2.4 The Racah-Casimir operators

Given the t-tensor of eq. (26), we define, for $su(n)$, the generalised Casimir operator of rank m by means of

$$C^{(m)} = t^{(m)}_{i_1 i_2 \dots i_m} X_{i_i} X_{i_2} \dots X_{i_m} \quad , \tag{39}$$

where the X_i are the generators of of the Lie algebra (2) of $su(n)$. The definition (39) produces each of the primitive $su(n)$ Casimir operators of orders $m \in \{2, 3, \dots, n\}$, and nothing else. This is so because the l $su(n)$ Lie algebra cohomology cocycles and their associated Omega tensors are in one-to-one correspondence with the t-tensors and hence with the $C^{(m)}$. Had we used the d-tensors in (39) instead of the t-tensors, we would always thereby obtain commuting $su(n)$ invariant operators, but of all orders for all $su(n)$ so that all but l of them are non-primitive. For low enough n, we can derive results which show explicitly how some of the non-primitive operators so obtained can be written in terms of primitive ones. But, in the context of the present work, this is not important: use of (39) bypasses the problem entirely.

We should point out one consequence of the uniformity in m of the definitions (27)–(30) of t-tensors. Eq. (27) implies

$$C^{(2)} = t^{(2)}_{ij} X_i X_j = n X_i X_i \quad , \tag{40}$$

with a possibly unexpected, but harmless, factor n. For example, for the $su(3)$ representation (λ, μ) in Dynkin coordinates, eq. (40) gives the eigenvalue

$$c^{(2)}(\lambda, \mu) = (\lambda^2 + \lambda\mu + \mu^2 + 3\lambda + 3\mu) \quad , \tag{41}$$

and, for the representation (λ, μ, ν) (*cf.* [7]) of $su(4)$

$$c^{(2)}(\lambda, \mu, \nu) = \frac{1}{2}(3\lambda^2 + 4\mu^2 + 3\nu^2 + 4\lambda\mu + 2\lambda\nu + 4\mu\nu + 12\lambda + 6\mu + 12\nu) \quad . \tag{42}$$

It may also be worth mentioning the result [21] for the eigenvalue of the cubic Casimir operator of $su(3)$

$$c^{(3)}(\lambda, \mu) = \frac{1}{6}(\lambda + 2\mu + 3)(2\lambda + \mu + 3)(\lambda - \mu) \quad . \tag{43}$$

One may use the defining representations of $su(3)$ and $su(4)$ to check that the normalisations of (41)–(43) give agreement with (39), (27) and (28).

3 The eigenvalues of the higher Casimir operators

For $su(n)$ for large enough n, $n \geq m$, we defined the m-th order Casimir operator by means of (39), where the X_i are the $su(n)$ generators. This yields an invariant operator $C^{(m)}(D)$ with eigenvalue $c^{(m)}(D)$ in any representation $X_i \mapsto D_i$ so that

$$C^{(m)}(D) = c^{(m)}(D)\, I_{\dim D} \quad . \tag{44}$$

Then, using (26), we find

$$\begin{aligned} c^{(m)}(D) \dim D &= \operatorname{Tr} C^{(m)}(D) \\ &= \Omega^{(2m-1)}_{i_1 j_i \ldots i_{m-1} j_{m-1} k_m} f_{i_1 j_1 k_1} \ldots f_{i_{m-1} j_{m-1} k_{m-1}} \operatorname{Tr} D_{k_1 \ldots k_m} \\ &= (-i)^{m-1}\, \Omega^{(2m-1)}_{i_1 j_i \ldots i_{m-1} j_{m-1} k_m} \operatorname{Tr} \left([D_{i_1}, D_{j_1}] \ldots [D_{i_{m-1}}, D_{j_{m-1}}] D_{k_m} \right) \\ &= (-2i)^{m-1}\, \Omega^{(2m-1)}_{i_1 j_i \ldots i_{m-1} j_{m-1} k_m} \operatorname{Tr} D_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k_m]} \quad . \end{aligned} \tag{45}$$

The first trace in (45), $\operatorname{Tr} D_{k_1 \ldots k_m}$ is in practice a unit weight symmetric trace $\operatorname{Tr} D_{(k_1} \ldots D_{k_m)}$ since $t^{(m)}$ is symmetric and $C^{(m)}(D)$ is given by (39) with $X = D$. For the last one we have written

$$D_{[ij \ldots s]} = D_{[i} D_j \ldots D_{s]} \quad . \tag{46}$$

We note here the transfer of the total antisymmetry from the Omega tensor to the trace of a product of $(2m-1)$ D's. This enables a crucial development since, as stated, $\operatorname{Tr}(D_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k_m]})$ must belong to the $(2m-1)$-cocycle space and, hence, has to be proportional to the only primitive, $SU(n)$-invariant, skewsymmetric, $(2m-1)$-Omega tensor.

Hence (see also subsection 3.1) for any representation D, we may write

$$\operatorname{Tr} D_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k_m]} = \left(\tfrac{1}{4} i\right)^{m-1} (gdi)^{(m)}(D)\, \Omega^{(2m-1)}_{i_1 j_i \ldots i_{m-1} j_{m-1} k_m} \quad , \tag{47}$$

thereby defining a quantity $(gdi)^{(m)}(D)$ which depends on m and D and in general on the Lie algebra $\mathfrak{g}$ in question. Since $\mathfrak{g} = su(n)$ here, $(gdi)^{(m)}(D)$ depends also on n. As noted above (gdi) is an acronym for generalised Dynkin index. Insertion of (47) into (45) immediately gives rise to one of the main results of this paper

$$\dim D\, c^{(m)}(D) = 2^{(1-m)} (gdi)^{(m)}(D) \Omega^{(2m-1)^2} \quad . \tag{48}$$

The importance of (48) is enhanced for $su(n)$ by the availability of the explicit result (5), valid for all n and for all m relevant to that n value, $m \leq n$. The relationship of our discussion of $c^{(m)}(D)$ and $(gdi)^{(m)}(D)$ to the work of previous authors is reviewed in the introduction. Our presentation conforms fully to this for $m = 2$. Otherwise our approach differs form that of others in view of the primary role in it that is played by the Omega tensors. This feature is inherited from [2], but (5) was not known when [2] was written.

We believe the analysis described here for the A_l family of Lie algebras extends to other classical compact simple algebras, exhibiting similar attractive features, and in some fashion to the exceptional algebras. For example, the crucial property of 'orthogonality' among two t-tensors of different order follows from their general definition in terms of their respective Ω cocycle tensors, and does not depend on the specific simple $\mathfrak{g}$ being considered [2]. However the corresponding tensor calculus, and the analogue of the $su(n)$ λ-matrix machinery is not yet developed sufficiently to produce simple expressions and formulas for all simple algebras.

3.1 Another approach to (47)

We sketch here another means of justifying our use of (47).

We have by now familiar steps

$$\mathrm{Tr}\, D_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k_m]} = \left(\tfrac{1}{2} i\right)^{m-1} f^{k_1}{}_{[i_1 j_1} \ldots f^{k_{m-1}}{}_{i_{m-1} j_{m-1}]} \mathrm{Tr}\, D_{k_1 \ldots k_m} \quad . \tag{49}$$

It is legitimate to insert round brackets first to enclose the set $k_1 \ldots k_{m-1}$ of indices, and then by use of the cyclic property of the trace to extend them to enclose the full set $k_1 \ldots k_m$. Now we may refer to the discussion in [2] for the construction of a basis for the vector space of $su(n)$-invariant symmetric tensors like $\mathrm{Tr}\, D_{(k_1 \ldots k_m)}$. The term in the expansion of $\mathrm{Tr}\, D_{(k_1 \ldots k_m)}$ with respect to this basis which involves $d^{(m)}_{(k_1 \ldots k_m)}$ is the significant one for our argument. Use of it immediately gives rise to a result for $\mathrm{Tr}\, D_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k_m]}$ of the form (47). All the other terms of the expansion are made up of symmetrised products of lower order d-tensors, and give rise to vanishing contributions to (47) in view of Jacobi identities, as shown in [5].

4 Application to certain classes of representations of $su(n)$

We consider here several important classes of representations of $su(n)$, including the fundamental ones, for which one may provide a definition that applies uniformly for all n.

4.1 The fundamental defining representation $\mathcal{F}$ of $su(n)$

The representation $\mathcal{F}$ is defined by (10) where the Gell-Mann lambda-matrices [13] are subject to

$$\mathrm{Tr}\, \lambda_i = 0 \quad , \quad \mathrm{Tr}\, \lambda_i \lambda_j = 2\delta_{ij} \quad , \quad {\lambda_i}^\dagger = \lambda_i \quad , \tag{50}$$

$$\lambda_i \lambda_j = \tfrac{2}{n} \delta_{ij} + (d + if)_{ijk} \lambda_k \quad , \tag{51}$$

valid for all $su(n)$, $n \geq 3$.

Using notation like that defined by (46), we quote from [5] the result for the trace of the fully antisymmetric product of an *odd* number of $(2m-1)$ lambda matrices

$$\mathrm{Tr}\, \lambda_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k]} = 2 i^{m-1} \Omega_{[i_1 j_1 \ldots i_{m-1} j_{m-1} k]} \quad . \tag{52}$$

We may now use (10) to substitute $\mathcal{F}_i$ for λ_i in (52), and deduce from (45), that

$$n c^{(m)}(\mathcal{F}) = 2^{(-m+1)} \Omega^{(2m-1)^2} \tag{53}$$

so that (48) gives

$$(gdi)^{(m)}(\mathcal{F}) = 1 \quad , \tag{54}$$

which also follows by comparing (52) and (47).

This result applies to all $su(n)$ and for any $m \leq n$. Eqs. (53) and the explicit expression (5) for $\Omega^{(2m-1)^2}$ show that $(gdi)^{(m)}(\mathcal{F})$ is zero otherwise. Eq. (54) does not itself provide new information (see table 2 of [9]) but (53) presents its information in a way that perhaps is. In this context, it may be worthwhile to display some formulas for eigenvalues in full detail

$$\begin{aligned} c^{(2)}(\mathcal{F}) &= \tfrac{1}{2}\,(n^2-1) \quad , & (55)\\ c^{(3)}(\mathcal{F}) &= \tfrac{1}{12}\,(n^2-1)(n^2-4) \quad , & (56)\\ c^{(4)}(\mathcal{F}) &= \tfrac{1}{180}\,(n^2-1)(n^2-4)(n^2-9) \quad , & (57)\\ c^{(5)}(\mathcal{F}) &= \tfrac{1}{1680}\,(n^2-1)(n^2-4)(n^2-9)(n^2-16) \quad . & (58) \end{aligned}$$

The factors that make $c^{(m)}(\mathcal{F})$ vanish when $m > n$ are visible here: the last factor above is $(n^2 - (m-1)^2)$, and a non-zero result requires $n \geq m$.

4.2 The adjoint representation ad of $su(n)$

The adjoint representation ad of the $su(n)$ algebra is defined by means of

$$X_i \mapsto ad_i \quad , \quad (ad_i)_{jk} = -i f_{ijk} \quad . \tag{59}$$

We do not possess a general formula for the factor μ that occurs in

$$\mathrm{Tr}\, ad_{[i_1 \ldots i_{2m-1}]} = \mu\, \Omega_{[i_1 \ldots i_{2m-1}]} \quad . \tag{60}$$

However it is easy to prove that for m *odd* the trace in (60) vanishes, so that $c^{(m)}(ad) = 0$ for all odd m. Since the matrices of the adjoint representation of any simple Lie algebra are antisymmetric, we find *e.g.* that

$$\mathrm{Tr}\, ad_{[ijkpq]} = \mathrm{Tr}\,(ad_{[ijkpq]})^T = -\mathrm{Tr}\, ad_{[qpkji]} = -\mathrm{Tr}\, ad_{[ijkpq]} = 0\,. \tag{61}$$

For m *even* no such conclusion follows: the same steps applied to, say, the sevenfold trace do not give zero, because now an odd permutation is required at the last step to restore the indices to their original order.

It remains to look at the even cases $m = 2, 4$ and $m = 6$, each by a separate calculation to get explicit formulas for $c^{(m)}(ad)$ for $su(n)$. The results are

$$\begin{aligned} c^{(2)}(ad) &= n^2 \quad , & (62)\\ c^{(4)}(ad) &= \tfrac{2^3}{6!}\, n^2(n^2-4)(n^2-9) \quad , & (63)\\ c^{(6)}(ad) &= \tfrac{2^5}{10!}\, n^2 \prod_{k=2}^{5}(n^2-k^2) \quad , & (64) \end{aligned}$$

from which we may conjecture that, for arbitrary *even* p,

$$c^{(p)}(ad) = \frac{2^{p-1}}{[2(p-1)]!}\, n^2 \prod_{k=2}^{p-1}(n^2-k^2) \quad , \tag{65}$$

whereas $c^{(odd)}(ad) = 0$.

The generalised Dynkin indices are then

$$\begin{aligned} (gdi)^{(2)}(ad) &= 2n \quad , & (66)\\ (gdi)^{(3)}(ad) &= 0 \quad , & (67)\\ (gdi)^{(4)}(ad) &= 2n \quad , & (68)\\ (gdi)^{(5)}(ad) &= 0 \quad , & (69)\\ (gdi)^{(6)}(ad) &= 2n \quad , & (70) \end{aligned}$$

etc. We recall that $\mathcal{C}^{(6)}$ is absent for $n < 6$ (eq. (64) contains explicit factors that reflect this), and hence note that (70) really only applies when $n \geq 6$. See also remarks that follow (103). Results (66) and (68) agree with results in [16].

The proof of (62) is easy. To obtain (63) we use

$$\mathrm{Tr}\, ad_{[ijklpqr]} = \left(\tfrac{1}{2}i\right)^3 f^a{}_{[ij} f^b{}_{kl} f^c{}_{pq]} \mathrm{Tr}(ad_r\, ad_{(abc)}) = \left(\tfrac{1}{4}i\right)^3 2n\, \Omega_{ijklpqr} \quad . \tag{71}$$

To perform the last step, a result from [2] is employed

$$\mathrm{Tr}\ \left(ad_r\, ad_{(abc)}\right) = \tfrac{n}{4} d^{(4)}_{(abcr)} + 2\delta_{(ab}\delta_{cr)} \quad . \tag{72}$$

In fact the second term of (72) does not contribute to (71) because it is non-primitive.

The proof of (64) similarly requires the formula

$$\mathrm{Tr}\, ad_{(abcder)} = \tfrac{n}{16}\, d^{(6)}_{(abcder)} + \ldots \quad , \tag{73}$$

where the dots denote terms which do not contribute to (64). One obtains this result by a method similar to that sketched in [2] to derive (A21) there. This requires a preliminary result

$$\mathrm{Tr}\, ad_{(abcd} D_{e)} = \tfrac{n}{8}\, d^{(5)}_{(abcde)} + \delta_{(ab} d_{cde)} \quad , \tag{74}$$

where $(D_i)_{jk} = d_{ijk}$. The deduction of each of the last two results entails a considerable amount of effort, making liberal use of identities found in the appendix to [2].

4.3 The reducible representation $\mathcal{D}$

The representation $\mathcal{D}$ of $\mathfrak{g} = su(n)$, of dimension $2^{[\dim \mathfrak{g}/2]} = 2^{[(n^2-1)/2]}$, has been described in the introduction. However in this case again, we lack an explicit analogue of (52). Again too the odd order Casimir operators have zero eigenvalues. To show this is true, we note there exists a matrix C such that

$$C\gamma_i C^{-1} = \pm {\gamma_i}^T \quad , \tag{75}$$

with the sign depending on $\dim \mathfrak{g}$. Hence in general $(S_i \equiv \mathcal{D}(X_i))$

$$S_i = -\tfrac{1}{4} i f_{ijk}\gamma_j\gamma_k \quad , \quad [S_i, S_j] = f_{ijk} S_k \quad , \tag{76}$$

obeys

$$S_i^T = -C S_i C^{-1} \quad . \tag{77}$$

This is sufficient to allow steps like those used in (60) to complete the demonstration, since the matrix C is invisible within the trace. To get non-vanishing results we look at m even, this time confining ourselves to the cases $m = 2$ and $m = 4$. We have

$$c^{(2)}(\mathcal{D}) = \tfrac{n}{8} {\Omega^{(3)}}^2 \tag{78}$$

$$c^{(3)}(\mathcal{D}) = 0 \tag{79}$$

$$c^{(4)}(\mathcal{D}) = -\tfrac{n}{64} {\Omega^{(7)}}^2 \quad , \tag{80}$$

and hence

$$(gdi)^{(2)}(\mathcal{D}) = \tfrac{n}{4}(\dim \mathcal{D}) \tag{81}$$

$$(gdi)^{(3)}(\mathcal{D}) = 0 \tag{82}$$

$$(gdi)^{(4)}(\mathcal{D}) = -\tfrac{n}{8}(\dim \mathcal{D}) \quad . \tag{83}$$

We have already noted that $\mathcal{D}$ is a direct sum of $2^{[(n-1)/2]}$ copies of the irreducible representation $\delta = (1,\ldots,1)$ of $su(n)$ [15], and that $\dim\mathcal{D} = 2^{[(n^2-1)/2]}$. It follows that the indices given by (81) and (83) are in all cases integers.

We remark also that the results (78)–(80) apply also to the representation δ of $su(n)$ since $\mathcal{D}$ is a direct sum of copies of δ.

For $su(3)$, for which δ coincides with the adjoint representation, $\mathcal{D}$ comprises two copies of ad, $c^{(2)}(\mathcal{D}) = c^{(2)}(\delta) = c^{(2)}(ad) = 9$ (eq. (62)), but

$$(gdi)^{(2)}(\mathcal{D}) = 2(gdi)^{(2)}(ad) \quad , \tag{84}$$

as is to be expected since by eq. (47) the dimension of the representation enters into the definition of the Dynkin index.

The easier of the proofs known to us for (80) follows the same lines as the proof of (62). It therefore requires the result

$$\mathrm{Tr}\,\left(S_d S_{(abc)}\right) = -\tfrac{n}{64} d^{(4)}_{(abcd)}(\dim\mathcal{D}) + \tfrac{3n^2-8}{64}\delta_{(ab}\delta_{cd)}(\dim\mathcal{D}) \quad , \tag{85}$$

which is proved in much the same way as (A.11) in [2] is proved. Some non-trivial work on traces of gamma matrices is involved. Also, as for (72), the second term of (85) does not contribute to the derivation of (83). The minus sign in (83) may be noted. It is not exceptional: the tables of [9] have plenty of negative entries.

4.4 The representations $\mathcal{S}_p$ of highest weight $(p, 0, \ldots, 0)$

The representations $\mathcal{S}_p$ carried by totally symmetric $su(n)$ tensors of rank p, the defining representation $\mathcal{F}$ being the case with $p = 1$, *i.e.* $\mathcal{S}_1 = \mathcal{F} = \mathcal{F}^1$. We can extend the results obtained for $\mathcal{F} = (1, 0, \ldots, 0)$ easily to $\mathcal{S}_2$ for which we define matrices

$$(M_i)_{\,a_1a_2, b_1b_2} = \delta_{(a_1}{}^{(b_1}\lambda_{i\,a_2)}{}^{b_2)} \quad . \tag{86}$$

It is easy to check that (86) satisfies (2). As previous sections indicate, what one needs for the calculation of generalised Dynkin indices in our approach is the evaluation of traces

$$\mathrm{Tr}\, D_{(a}D_b \ldots D_{s)} \quad . \tag{87}$$

It is easy to use (86) to compute

$$\mathrm{Tr}\, M_i M_j = \tfrac{1}{2}(n+2)\delta_{ij} \tag{88}$$
$$\mathrm{Tr}\, M_{(i}M_j M_{k)} = \tfrac{1}{4}(n+4)d_{ijk} \tag{89}$$
$$\mathrm{Tr}\, M_{(i}M_j M_k M_{l)} = \tfrac{1}{8}(n+8)d^{(4)}_{(ijkl)} + \ldots \quad , \tag{90}$$

where the dots indicate lower order terms known but known also, because of Jacobi identities, not to contribute to the calculation of the eigenvalues $c^{(4)}(\mathcal{S}_2)$. Thus we find that all results agree with

$$(gdi)^{(m)}(\mathcal{S}_2) = n + 2^{m-1} \quad . \tag{91}$$

To proceed further it is advisable to use heavier duty methods. Bird-track methods allow us to subsume the calculations just done into the treatment of the general p case, by dealing with the totally symmetrised p-fold direct products of defining representations.

Our results include the following

$$
\begin{aligned}
(gdi)^{(m)}(\mathcal{S}_p) &= \frac{(n+p)!}{(p-1)!(n+1)!} \quad , \quad m=2 \quad ; && (92)\\
&= \frac{(n+p)!}{(p-1)!(n+2)!}(n+2p) \quad , \quad m=3 \quad ; && (93)\\
&= \frac{(n+p)!}{(p-1)!(n+3)!}(n^2-n+6pn+6p^2) \quad , \quad m=4 \quad . && (94)
\end{aligned}
$$

The result (91) for $\mathcal{S}_2$ of course conforms to these results. To derive these expressions, we employ results for totally symmetrised traces that appear in [16] as eqs. (12.69)–(12.71). We note also from [16] the diagram (12.64) used to define the matrices of $(p,0,\ldots,0)$, and the essential results (5.19) and (5.23) given in the valuable chapter in [16] on permutations.

4.5 The l fundamental representations $\mathcal{F}^s$ of $su(n)$

The representation $\mathcal{F}^2 = (0,1,0,\ldots,0)$ of $su(n)$ is the antisymmetric part of the direct product $\mathcal{F}\otimes\mathcal{F}$, where $\mathcal{F}=\mathcal{F}^1$ is the defining representation of $su(n)$. Thus we define for $\mathcal{F}^2$ the matrices

$$
(N_i)_{\,a_1a_2,b_1b_2} = \delta_{[a_1}{}^{[b_1}\lambda_{i\,a_2]}{}^{b_2]} \quad . \qquad (95)
$$

This differs form (86) only in that round symmetrisation brackets are replaced by square antisymmetrisation brackets. We find the following results

$$
\begin{aligned}
\mathrm{Tr}\, N_iN_j &= \tfrac{1}{2}(n-2)\delta_{ij} && (96)\\
\mathrm{Tr}\, N_{(i}N_jN_{k)} &= \tfrac{1}{4}(n-4)d_{ijk} && (97)\\
\mathrm{Tr}\, N_{(i}N_jN_kN_{l)} &= \tfrac{1}{8}(n-8)d^{(4)}_{(ijkl)} + \ldots \quad . && (98)
\end{aligned}
$$

Again all these results agree with the general statement

$$
(gdi)^{(m)}(\mathcal{F}^2) = n - 2^{m-1} \quad . \qquad (99)
$$

It is of clear interest to proceed further down the antisymmetrisation path. For $su(n)$, the totally antisymmetrised parts of the s-fold products of defining representations correspond to the fundamental representations $\mathcal{F}^s$ of $su(n)$ for $s=1,2,\ldots l=(n-1)$, *i.e.* $\mathcal{F}^s$ has a one in the s-th place of its Dynkin coordinate description and zeros elsewhere: its highest weight is the s-th fundamental dominant weight.

Using bird-track methods, we find

$$
\begin{aligned}
(gdi)^{(m)}(\mathcal{F}^s) &= \frac{(n-2)!}{(s-1)!(n-s-1)!} \quad , \quad m=2 \quad ; && (100)\\
&= \frac{(n-3)!}{(s-1)!(n-s-1)!}(n-2s) \quad , \quad m=3 \quad ; && (101)\\
&= \frac{(n-4)!}{(s-1)!(n-s-1)!}(n^2+n-6sn+6s^2) \quad , \quad m=4 \quad . && (102)
\end{aligned}
$$

The result (99) for $s=2$ conforms to these results; for s=1 we get $(gdi)^{(m)}(\mathcal{F}^1)$=1, which also follows from (92)–(94) for p=1 as it should. Results analogous to (100)–(102) for $m=5$ and $m=6$ have also been computed. For $s=2$ they each agree with (99). When $s=3$ we have

$$
(gdi)^{(5)}(\mathcal{F}_3) = \tfrac{1}{2}(n-6)(n-27) \quad . \qquad (103)
$$

Since $su(n)$ has a fifth order Casimir operator only for $n \geq 5$, (103) applies only for such n. It gives 11 for $n = 5$ and vanishes for $n = 6$, but is non-zero for all larger n except $n = 27$. In other words $c^{(5)}(\mathcal{F}_3)$ vanishes when $n = 6$ in virtue of its (gdi) factor rather than its Ω factor.

To obtain these results we have followed methods for the antisymmetric case analogous to those of the previous section for the symmetric case. Permutation lemmas (5.19) and (5.23) of [16] expedite the work.

The results of this section permit the evaluation of all the indices of all the fundamental representations of $su(n)$ for all $n \leq 6$. These are presented in tables in Sec. 5.

The results of (92)–(94) are very closely related to results to be found in chapter 16 of [16]. Although no such statement holds for (100)–(102), all the tools needed to derive them were found in [16] polished and ready for use.

5 Tables of indices for $su(n)$ for $n \leq 6$

The (gdi) indices presented in the tables that follow have been deduced from (99)–(103) one easy check is available. If $X_i \mapsto D_i$ defines the representation D of $su(n)$, then

$$X_i \mapsto \bar{D}_i = -{D_i}^T \tag{104}$$

defines the representation $\bar{D}$ of $su(n)$. It then follows that

$$c^{(m)}(D) = \pm c^{(m)}(\bar{D}) \quad , \tag{105}$$

where the plus applies to even m and the minus to m odd. The data in the tables below conforms to this. Further some entries for $su(4)$ and $su(6)$ agree with the consequence of (105) that odd Casimir operators have zero eigenvalues for self-conjugate representations.

Generalised Dynkin indices of $su(3)$		
$su(3)$	$s=1$ or (1,0)	$s=2$ or (0,1)
$m=2$	1	1
$m=3$	1	-1

Generalised Dynkin indices of $su(4)$			
$su(4)$	$s=1$ or (1,0,0)	$s=2$ or (0,1,0)	$s=3$ or (0,0,1)
$m=2$	1	2	1
$m=3$	1	0	-1
$m=4$	1	-4	1

Generalised Dynkin indices of $su(5)$				
$su(5)$	$s=1$ or (1,0,0,0)	$s=2$ or (0,1,0,0)	$s=3$ or (0,0,1,0)	$s=4$ or (0,0,0,1)
$m=2$	1	3	3	1
$m=3$	1	1	-1	-1
$m=4$	1	-3	-3	1
$m=5$	1	-11	11	-1

Generalised Dynkin indices of $su(6)$					
$su(6)$	(1,0,0,0,0)	(0,1,0,0,0)	(0,0,1,0,0)	(0,0,0,1,0)	(0,0,0,0,1)
$m = 2$	1	4	6	4	1
$m = 3$	1	2	0	-2	-1
$m = 4$	1	-2	-6	-2	1
$m = 5$	1	-10	0	10	-1
$m = 6$	1	-26	66	-26	1

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