

Lie's, Tracks, andape;
classical and exceptional

Problem: nature loves continuous symmetries

$SO(3)$: QM spins

$SU(3), \dots, E_8(248)$: elementary particles

Setting: Lie algebras

Wigner-Eckart the, $3n-j$ coefficients

Solution: Cartan-Killing ✓

YAS: primitive invariants

$SU(n), SO(n), Sp(n)$ hum-drum ✓

$E_8, \dots$ cute

Magre: negative dimensions

$SO(n) = Sp(-n)$

Spinors = spinsters

$E_7(56) = SO(4)$

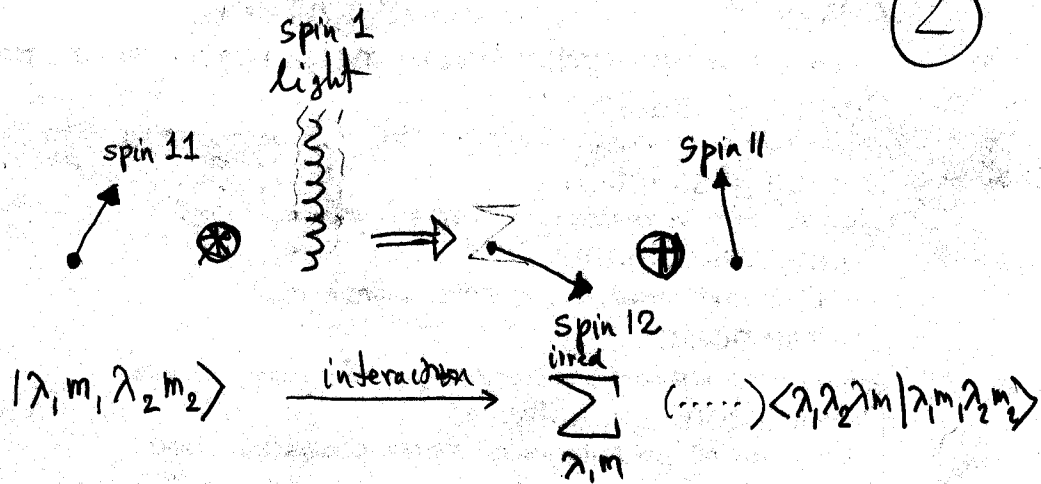
Magic Triangle

Problem: MOALA

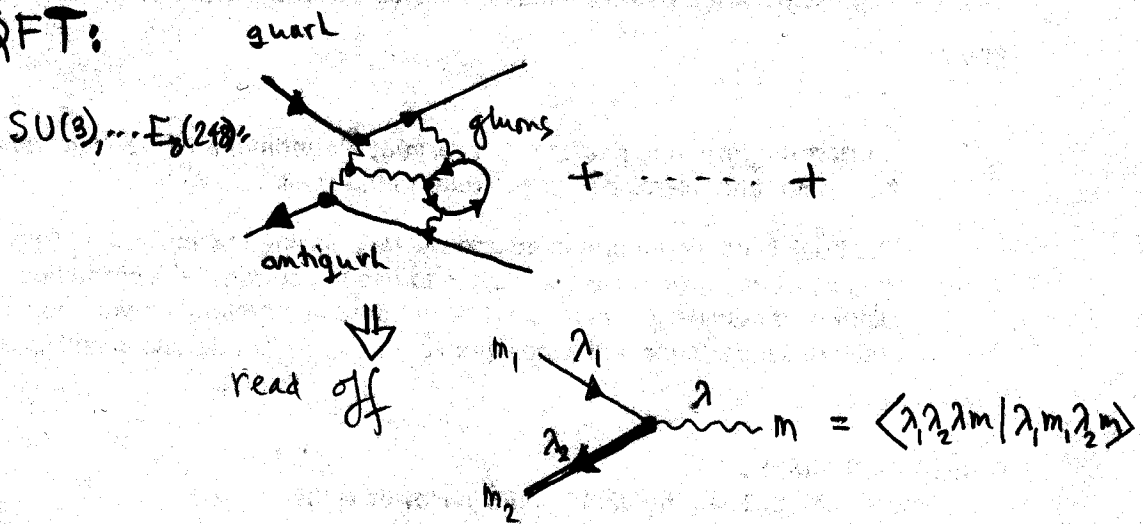
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1) Problem

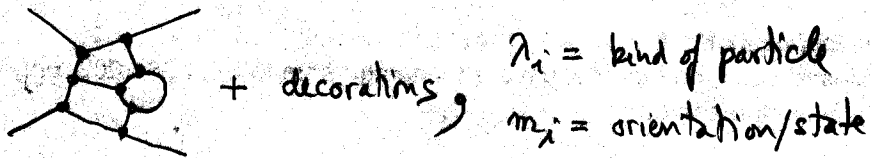
a) QM:
SO(3)
rotations



b) QFT:



trivalent graphs:



c) ~~what~~ what continuous symmetries exist?

~~compute~~ compute ?

(3)

Setting

Groups

vectors

$$X_a \in V, \quad X^a \in \overline{V}$$

tensors

$$X_{a_1 a_2 \dots a_p}^{b_1 b_2 \dots b_p} \in V^p \otimes \overline{V}^q$$

$n = \dim(V)$, $\overline{V}$ is
addition, scalar multiplication,
inner product $\langle y, x \rangle = X_a y^a$

(state with several particles)

group

$$G \times G \rightarrow G$$

multiplication, inverses, identity

representation, linear

$$X'_a = G_a^b X_b, \quad X'^a = (G^+)^a_b X^b$$

$$X'^{a_1 a_2 \dots a_p}_{b_1 b_2 \dots b_p} = G_{b_1}^{c_1} \dots G_{b_p}^{c_p} (G^+)^{a_1}_{d_1} \dots (G^+)^{a_p}_{d_p} X^{d_1 \dots d_p}_{c_1 \dots c_p}$$

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Lie's

continuous, connected to identity, (compact)

$$G = e^{i\epsilon_i T_i} \approx 1 + i\epsilon_i T_i$$

$$b \rightarrow a = b \rightarrow a + i \sum_{j=1}^N \epsilon_j \rightarrow j$$

$$\underline{G \times G \rightarrow G}$$

Lie algebra $[T_i, T_j] = \sum_k C_{ij}^k T_k$

$$\cancel{V} - \cancel{X} = Y$$

Jacobi id

SOLUTION

Killing: $SU(2)$ $T_i = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}$

any Lie $= \{E^*, H, E\}$
 $\begin{matrix} \nearrow & \uparrow & \nwarrow \\ \text{lower} & \text{diagonal,} & \text{raise} \\ & \# = \text{rank } G & \end{matrix}$

$\Rightarrow$ finite lattice

Cartan-Killing $\Rightarrow$ complete classification

$A_r, B_r, C_r, D_r; G_2, F_4, E_6, E_7, E_8$

A beautiful solution (can impede progress for many decades)

PROBLEMS:

* what do $G_2, \dots, E_8$ mean?

* better way to compute 

YAS: geometry $\iff$ invariants

1) length $\vec{x}^2 = x_i \delta_{ij} x_j$

Volume $\vec{x} \wedge \vec{y} \wedge \vec{z} = \epsilon_{ij\cdots k} x_i y_j \cdots z_k$

$SO(n) \iff \delta_{ij}, \epsilon_{ij\cdots k}$

2) magnitude $\overline{x}^2 = X^* X = X^a \delta_a^b X_b$

$U(n) \iff \delta_b^a$

3) sesquilinear inv

$x_i f_{ij} y_j, f_{ij} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$Sp(2m) \iff f_{ij}$

4) cubic invariant

$x_i x_j x_k C_{ijk}, C_{ijk} = -C_{jik} = -C_{ikj}$

$E_8, E_7, \dots \iff \delta_{ij}, C_{ijk}$

5) quartic invariant

E_7

6) higher ???

IAS = Primitive Invariants

①

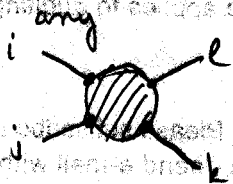
invariant tensors:

$$G \cdot g = g \quad (g \in V^p \otimes \nabla^q \text{ (+index decorations)})$$

SO(3) example: $g = \{\delta_{ij}, \epsilon_{ijk}\}$

primitive invariant tensors;

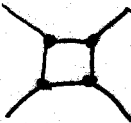
any trees



$$= A \begin{matrix} i & j \\ \text{---} & \text{---} \\ k & l \end{matrix} + B \begin{matrix} i & j \\ \text{---} & \text{---} \\ k & e \end{matrix} + C \begin{matrix} i & j \\ \text{---} & \text{---} \\ k & l \end{matrix} + D \begin{matrix} i & j \\ \text{---} & \text{---} \\ k & l \end{matrix} + E \begin{matrix} i & j \\ \text{---} & \text{---} \\ k & l \end{matrix} + F \begin{matrix} i & j \\ \text{---} & \text{---} \\ k & l \end{matrix}$$

$$= A \delta_{ij} \delta_{kl} + \dots + D \epsilon_{ijm} \epsilon_{mkl} + \dots$$

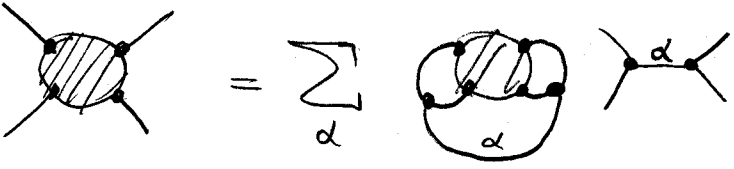
loops:



= not primitive, reducible

Schur's Lemma, Wigner-Eckhart theorem, . . .

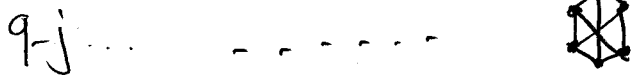
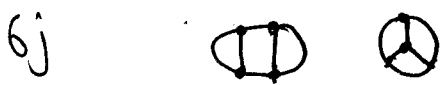
example:



n -legs = (vacuum bubble) (kinematic)

need only vacuum bubbles

Wigner $3n-j$ coefficients

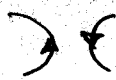


YAS: invariants $\rightarrow$ compute all vacuum bubbles, (9)

boring example: $SU(n)$

pp 49-152

$$V \otimes \bar{V} \rightarrow A$$

defining
(dual)  =  - $\frac{1}{n}$ 

$$(T_i)_a^b (T_i)_c^d = \delta_a^d \delta_b^c - \frac{1}{n} \delta_a^b \delta_c^d$$

all bubbles
✓ can be reduced
to polyhedra in 4

example:

$$\text{Diagram 1} = \sum \text{Diagram 2} = 2n^2(n^2 + 12) \text{ ---}$$


instructive example: E_8 family

~~primitives~~
primitives: $\{ \text{---}, \text{---} \}$ (ie' no $\times$!)

1. Jacobi relation, symmetry, ...

$$A \otimes A \rightarrow A \otimes A$$

only new inv. tensor (matrix of dim $[N^2 \times N^2]$)

$$Q_{ij,kl} = \begin{array}{c} i \text{---} \bullet \text{---} l \\ | \quad | \\ j \text{---} \bullet \text{---} k \end{array} = \sum_e C_{ine} C_{jke}$$

by assumption,  reducible

$$0 = \left\{ \begin{array}{c} \text{---} \bullet \text{---} \bullet \text{---} \\ | \quad | \quad | \\ \text{---} \bullet \text{---} \bullet \text{---} \end{array} + p \begin{array}{c} \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \end{array} + q \begin{array}{c} \text{---} \bullet \text{---} \\ | \\ \text{---} \bullet \text{---} \end{array} \right\} P_{\text{symmetric}} \quad (\text{antisym. reduced by Jacobi rel.})$$

$$0 = (Q^2 + pQ + q1)$$

quadratic equation! symmetry fixes p, q , get

$$Q^2 - \frac{1}{6}Q - \frac{5}{3(N+2)} = 0 \quad (\text{train calculator})$$

$$\text{eigenvalues: } (Q - \lambda)(Q - \lambda^*) = 0$$

split sym rep of $SO(N)$ into two subspaces

$$P_{\square} = \frac{Q - \lambda^* 1}{\lambda^* - \lambda}, \quad P_{\blacksquare} = \frac{Q - \lambda 1}{\lambda - \lambda^*}, \quad P_{\alpha}^2 = P_{\alpha}, \quad P_{\alpha} P_{\beta} = 0 \quad \alpha \neq \beta$$

Diophantine conditions

dim of the group is an integer, positive (?)

$$N+2 = \frac{5}{3\lambda(\lambda-1/6)}$$

convenient reparam. $\lambda = -\frac{1}{m-6}$

$$N = -122 + 10m + \frac{360}{m} \quad \text{only 27 rational } m \text{ solutions}$$

$d_{\blacksquare}$ is integer (irrep of E_8)

$$d_{\blacksquare} = \text{tr } P_{\blacksquare} = \left(\begin{array}{c} \text{vacuum} \\ \text{bubble} \end{array} \right)$$

$$= \frac{270(m-6)^2(5m-36)(2m-9)}{m(m+6)}$$

fewer m solve this

next

$$A \otimes A \otimes A \rightarrow A \otimes A \otimes A$$

$$\tilde{Q} = \begin{array}{c} \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \\ | \quad | \\ \text{---} \bullet \text{---} \end{array}$$

(nothing else!)

inherited from $A \otimes A$ decomposition

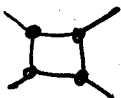
$$0 = (\tilde{Q} - \frac{1}{18})(\tilde{Q} - \lambda)(\tilde{Q} - \lambda^*)$$

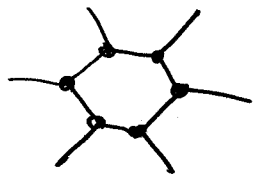
now so restrictive that integer dimension of reps eliminates all m but those identifiable as

$$\{A, A_2, G_2, D_4, F_4, E_6, E_7, E_8\}$$

(PC 1979)

classification $\leftrightarrow$ Cartan-Killing
but

computation:  $\checkmark$



8-loop $\dots = ?$

(except for G_2 , solved $\checkmark$)

need more than pencil, paper and
Göteborg $\leftrightarrow$ København train?

MAGIC I

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negative dimensions

for vacuum bubbles

$$SO(n) = Sp(-n)$$

$$\text{spinors} = \text{spinsters}$$

$$E_7(56) = SO(4)$$

$$\dots = \dots$$

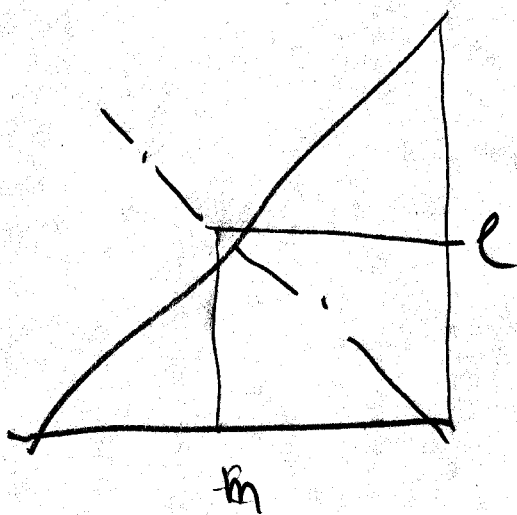
$$(R. Penrose \quad SO(2) = Sp(-2))$$

(still open)

Exceptional magic

$$N = \frac{(l-6)(m-6)}{3} - 72 + \frac{360}{e} + \frac{360}{m}$$

all results on exceptional algebras



why?

why symmetric

MOALA speculation

$\exists$ set of special functions

$$\pi(u, v) \in \mathbb{C}$$

such that $\pi(m, e) \in \mathbb{N}$

Mother of all Lie Algebras

Vogel, Deligne, De Mann, Landsberg (1995+..)

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comments

↙ new, not in $SO(N)$

$$(*) A \circ A = \square + \begin{array}{|c|} \hline \square \\ \hline \end{array} + \bullet + \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} + \blacksquare$$

is "uniform" because $Q = \begin{array}{|c|} \hline \bullet \\ \hline \end{array} \begin{array}{|c|} \hline \bullet \\ \hline \end{array}$ is the only new invariant

$$(*) A \circ A \circ A$$

is still linear in m , i.e., no factors like

$$d_n = \frac{(\dots)(m^2 + 2m - 7)(\dots)}{(\dots)}$$

because $\otimes^3 A$ decomposition still depends on $\otimes^2 A$ i.e. λ .

However

→ de Man computed $\otimes^4 A$ (from Cartan-Killing const)

still linear

→ Landsberg computed one of the reps in $\otimes^k A$ — 11 —

$\otimes^k A$

(impressive!)

still linear

puzzle!