9.2 Birdtracks - updated history

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Young tableaux and (non-Hermitian) Young projection operators were introduced by Young [21] in 1933 (Tung monograph [20] is a standard exposition). In 1937 R. Brauer [4] introduced diagrammatic notation for δ_{ij} in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. R. Penrose's papers were the first to cast the Young projection operators into a diagrammatic form. In 1971 monograph [14] Penrose introduced diagrammatic notation for symmetrization operators, Levi-Civita tensors [16], and "strand networks" [13]. Penrose credits Aitken [2] with introducing this notation in 1939, but inspection of Aitken's book reveals a few Brauer diagrams for permutations, and no (anti)symmetrizers. Penrose's [15] 1952 initial ways of drawing symmetrizers and antisymmetrizers are very aesthetical, but the subsequent developments gave them a distinctly ostrich flavor [15]. In 1974 G. 't Hooft introduced a double-line notation for U(n) gluon group-theory weights [1]. In 1976 Cvitanović [8] introduced analogous notation for SU(N), SO(n) and Sp(n). For several specific, few-index tensor examples, diagrammatic Young projection operators were constructed by Canning [6], Mandula [12], and Stedman [18].

The 1975–2008 Cvitanović diagrammatic formulation of the theory of all semi-simple Lie groups [9] as a way to compute group theoretic wights without any recourse to symbols goes conceptually and profoundly beyond the Penrose notation (indeed, Cvitanović "birdtracks" bear no resemblance to Penrose's "fornicating ostriches" [15]).

A chapter in Cvitanović 2008 monograph [9] sketches how birdtrack (diagrammatic) Young projection operators for arbitrary irreducible representation of SU(N) could be constructed (this text is augmented by a 2005 appendix by Elvang, Cvitanović and Kennedy [10] which, however, contains a significant error). Keppeler and Sjödahl [11] systematized the construction by offering a simple method to construct Hermitian Young projection operators in the birdtrack formalism. Their iteration is easy to understand, and the proofs of Hermiticity are simple. However, in practice, the algorithm is inefficient - the expression balloon quickly, the Young projection operators soon become unwieldy and impractical, if not impossible to implement.

The Alcock-Zeilinger algorithm, based on the simplification rules of ref. [3], leads to explicitly Hermitian and drastically more compact expressions for the projection operators than the Keppeler-Sjödahl algorithm [11]. Alcock-Zeilinger fully supersedes Cvitanović's formulation, and any future full exposition of reduction of SU(N) tensor products into irreducible representations should be based on the Alcock-Zeilinger algorithm.

References

[1] G. 't Hooft, "A planar diagram theory for strong interactions", Nucl. Phys. B **72**, 461–473 (1974).

[2] A. Aitken, *Determinants & Matrices* (Oliver & Boyd, Edinburgh, 1939).

[3] J. Alcock-Zeilinger and H. Weigert, "Compact Hermitian Young projection operators", J. Math. Phys. **58**, 051702 (2017).

[4] R. Brauer, "On algebras which are connected with the semisimple continuous groups", Ann. Math. **38**, 857 (1937).

[5] N. B. Budanur and P. Cvitanović, "Unstable manifolds of relative periodic orbits in the symmetry-reduced state space of the Kuramoto-Sivashinsky system", J. Stat. Phys. **167**, 636–655 (2015).

[6] G. P. Canning, "Diagrammatic group theory in quark models", Phys. Rev. D **18**, 395–410 (1978).

[7] J.-Q. Chen, J. Ping, and F. Wang, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).

[8] P. Cvitanović, "Group theory for Feynman diagrams in non-Abelian gauge theories", Phys. Rev. D **14**, 1536–1553 (1976).

[9] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[10] H. Elvang, P. Cvitanović, and A. Kennedy, "Diagrammatic Young projection operators for u(n)", J. Math. Phys. **46**, 043501 (2005).

[11] S. Keppeler and M. Sjödahl, "Hermitian Young operators", J. Math. Phys. **55**, 021702 (2014).

[12] J. E. Mandula, Diagrammatic techniques in group theory, Univ. of Southampton, Notes taken by S. N. Coulson and A. J. G. Hey, 1981.

[13] R. Penrose, "Angular momentum: An approach to combinatorical space-time", in *Quantum Theory and Beyond*, edited by T. Bastin (Cambridge Univ. Press, Cambridge, 1971).

[14] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial mathematics and its applications*, edited by D. J. J.A. Welsh (Academic, New York, 1971), pp. 221–244.

[15] R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe* (A. A. Knopf, New York, 2005).

[16] R. Penrose and M. A. H. MacCallum, "Twistor theory: An approach to the quantisation of fields and space-time", Phys. Rep. **6**, 241–315 (1973).

[17] D. S. Silver, "The new language of mathematics", Amer. Sci. **105**, 364 (2017).

[18] G. E. Stedman, *Diagram Techniques in Group Theory* (Cambridge U. Press, Cambridge, 1990).

[19] W.-K. Tung, *Group Theory in Physics* (World Scientific, Singapore, 1985).

13.1 Group theory news

Mathematicians map E_8, and it is bigger than the human genome.

Turns out, applications of group theory go way beyond what is covered in this course:

Group theory of defamation: The officers argued Sawant's statements impugned them individually even though she only spoke about the police department as a whole. The court says suing as individuals and advancing a group theory of defamation takes far more than the officers showed in their complaint.

> [W]hether proceeding under an individual or group theory, Plaintiffs must plead that the statements "specifically" identified or singled them out, or was understood as "referring to [them] in particular." Sims, 20 Wn. App. at 236.

16.1 Literature

We noted in sect. 2.1 that a practically-minded physicist always has been, and continues to be resistant to gruppenpest. Apparently already in 1910 James Jeans wrote, while discussing what should a physics syllabus contain: "We may as well cut out the group theory. That is a subject that will never be of any use in physics."

Voit writes here about the "The Stormy Onset of Group Theory in the New Quantum Mechanics," citing Bonolis [3] *From the rise of the group concept to the stormy onset of group theory in the New Quantum Mechanics. A saga of the invariant characterization of physical objects, events and theories*.

Chayut [4] *From the periphery: the genesis of Eugene P. Wigner's application of group theory to quantum mechanics* traces the origins of Wigner's application of group theory to quantum physics to his early work as a chemical engineer, in chemistry and crystallography. "In the early 1920s, crystallography was the only discipline in which symmetry groups were routinely used. Wigner's early training in chemistry exposed him to conceptual tools which were absent from the pedagogy available to physicists for many years to come. This both enabled and pushed him to apply the group theoretic approach to quantum physics. It took many years for the approach first introduced by Wigner in the 1920s – and whose reception by the physicists was initially problematical

– to assume the pivotal place it now holds." Another historical exposition is given by Scholz [9] *Introducing groups into quantum theory (1926–1930)*.

So what is group theory good for? By identifying the symmetries, one can apply group theory to determine good quantum numbers which describe a physical state (i.e., the irreps). Group theory then says that many matrix elements vanish, or shows how are they related to others. While group theory does not determine the actual value of a matrix element of interest, it vastly simplifies its calculation.

The old fashioned atomic physics, fixated on $\mathrm{SO}(3)$ / $\mathrm{SU}(2)$, is too explicit, with too many bras and kets, too many square roots, too many deliriously complicated Clebsch-Gordan coefficients that you do not need, and way too many labels, way too explicit for you to notice that all of these are eventually summed over, resulting in a final answer much simpler than any of the intermediate steps.

I wrote my book [6] *Group Theory - Birdtracks, Lie's, and Exceptional Groups* to teach you how to compute everything you need to compute, without ever writing down a single explicit matrix element, or a single Clebsch-Gordan coefficient. There are two versions. There is a particle-physics / Feynman diagrams version that is index free, graphical and easy to use (at least for the low-dimensional irreps). The key insights are already in Wigner's book [11]: the content of symmetry is a set of invariant numbers that he calls $3n$-j's. Then there are various mathematical flavors (Weyl group on Cartan lattice, etc.), elegant, but perhaps too elegant to be computationally practical.

But it is nearly impossible to deprogram people from years of indoctrination in QM and EM classes. The professors have no time to learn new stuff, and students love manipulating their mu's and nu's.

References

[1] S. L. Adler, J. Lieberman, and Y. J. Ng, "Regularization of the stress-energy tensor for vector and scalar particles propagating in a general background metric", Ann. Phys. **106**, 279–321 (1977).

[2] L. Bel, "Étude algébrique d'un certain type de tenseurs de courbure. Le cas 3 de Petrov", C. R. Acad. Sci., Paris **247**, 2096–2099 (1958).

[3] L. Bonolis, "From the Rise of the Group Concept to the Stormy Onset of Group Theory in the New Quantum Mechanics. A saga of the invariant characterization of physical objects, events and theories", Rivista Nuovo Cim. **27**, 1–110 (2005).

[4] M. Chayut, "From the periphery: the genesis of Eugene P. Wigner's application of group theory to quantum mechanics", Found. Chem. **3**, 55–78 (2001).

[5] C. D. Collinson, "A class of conserved tensors in an arbitrary gravitational field", Math. Proc. Cambridge Philos. Soc. **58**, 346–362 (1962).

[6] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

[7] C. Lanczos, "A remarkable property of the Riemann-Christoffel tensor in four dimensions", Ann. Math. (2) **39**, 842–850 (1938).

[8] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial mathematics and its applications*, edited by D. J. J.A. Welsh (Academic, New York, 1971), pp. 221–244.

[9] E. Scholz, "Introducing groups into quantum theory (1926–1930)", Hist. Math. **33**, 440–490 (2006).

[10] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[11] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).