9.1.1 Invariants

One constructs the irreps of finite groups by identifying matrices that commute with all group elements, and using their eigenvalues to decompose arbitrary representation of the group into a unique sum of irreps. The same strategy works for the compact Lie groups, (9.9), and is indeed the key idea that distinguishes the invariance groups classification developed in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [9] from the 19th century Cartan-Killing classification of Lie algebras.

Definition. Éhe Éector $q \in V$És Én *nvariant ector* Éf Éor Ény Éransformation $g \in \mathcal{G}$

$$q = Gq\,. \tag{9.6}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} = G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \ldots G_{b_1}{}^{d_1} \ldots G_{b_q}{}^{d_q} x^{c_1 c_2 \ldots c_p}_{d_1 \ldots d_q} \,. \tag{9.7}$$

If a bilinear form $m(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a{}^b = G_a{}^c G^b{}_d M_c{}^d \tag{9.8}$$

is an *invariant matrix*. Multiplying with $G_b{}^e$ and using the unitary, we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, \mathbf{M}] = 0 \,. \tag{9.9}$$

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations (9.7) that preserve the primitive invariant relations (and, by extension, *all* invariant relations)

$$\begin{aligned} p_1(x, \bar{y}) &= p_1(Gx, \bar{y}G^\dagger) \\ p_2(x, y, z, \ldots) &= p_2(Gx, Gy, Gz \ldots) \,, \qquad \ldots . \end{aligned} \tag{9.10}$$

Unitarity guarantees that all contractions of primitive invariant tensors, and hence all composed tensors $h \in H$, are also invariant under action of $\mathcal{G}$. As we assume unitary $\mathcal{G}$, it follows that the list of primitives must always include the Kronecker delta.

Example 2. If $p^a q_a$ is the only invariant of $\mathcal{G}$

$$p'^a q'_a = p^b (G^\dagger G)_b{}^c q_c = p^a q_a \,, \tag{9.11}$$

then $\mathcal{G}$ is the full *unitary group* $U(n)$ (invariance group of the complex norm $|x|^2 = x^b x_a \delta^a_b$), whose elements satisfy

$$G^\dagger G = 1 \,. \tag{9.12}$$

Example 3. If we wish the z-direction to be invariant in our 3-dimensional space, $q = (0, 0, 1)$ is an invariant vector (9.6), and the invariance group is $O(2)$, the group of all rotations in the x-y plane.

9.1.2 Discussion

Henriette Roux Please explain when one keeps track of the order of tensorial indices?

Predrag In a tensor, upper, lower indices are separately ordered - and that order matters. The simplest example: if some indices form an antisymmetric pair, writing them in wrong order gives you a wrong sign. In a matrix representation of a group action, one has to distinguish between the "in" set of indices – the ones that get contracted with the initial tensor, and the "out" set of indices that label the tensor after the transformation. Only if the matrix is Hermitian the order does not matter. If you understand Eq. (3.22) in birdtracks.eu, you get it. Does that answer your question?

9.1.3 Infinitesimal transformations, Lie algebras

A unitary transformation G infinitesimally close to unity can be written as

$$G_a{}^b = \delta_a^b + iD_a^b \,, \tag{9.13}$$

where D is a hermitian matrix with small elements, $|D_a^b| \ll 1$. The action of $g \in \mathcal{G}$ on the conjugate space is given by

$$(G^\dagger)_b{}^a = G^a{}_b = \delta_b^a - iD_b^a \,. \tag{9.14}$$

D can be parametrized by $N \leq n^2$ real parameters. N, the maximal number of independent parameters, is called the *dimension* of the group (also the dimension of the Lie algebra, or the dimension of the adjoint rep).

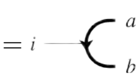
Here we shall consider only infinitesimal transformations of form $G = 1 + iD$, $|D_b^a| \ll 1$. We do not study the entire group of invariant transformation, but only the transformations connected to the identity. For example, we shall not consider invariances under coordinate reflections.

The generators of infinitesimal transformations (9.13) are hermitian matrices and belong to the $D_b^a \in V \otimes \bar{V}$ space. However, not any element of $V \otimes \bar{V}$ generates an allowed transformation; indeed, one of the main objectives of group theory is to define the class of allowed transformations.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V. We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin, straight line,

$$\delta_{ij} = i \text{ ———— } j \,, \qquad i, j = 1, 2, \ldots, N \,, \tag{9.15}$$

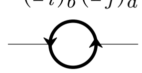
and the corresponding transformation generators by

$$(C_A)_i,{}_b^a = \frac{1}{\sqrt{a}}(T_i)_b^a = $$ 

$$a, b = 1, 2, \ldots, n$$
$$i = 1, 2, \ldots, N \,.$$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition

$$(T_i)_b^a (T_j)_a^b = \text{tr}\,(T_i T_j) = a\,\delta_{ij}$$

 $= a$ ———— . (9.16)

For every invariant tensor q, the infinitesimal transformations $G = 1 + iD$ must satisfy the invariance condition (9.6). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = \mathbf{P}_A H \in V \otimes \bar{V}$,

$$D_b^a = \frac{1}{a}(T_i)_b^a \epsilon_i \,, \tag{9.17}$$

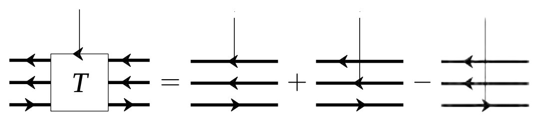
we obtain the *invariance condition* which the *generators* must satisfy: they *annihilate* invariant tensors:

$$T_i q = 0 \,. \tag{9.18}$$

To state the invariance condition for an arbitrary invariant tensor, we need to define the action of generators on the tensor reps. By substituting $G = 1 + i\epsilon \cdot T + O(\epsilon^2)$ and keeping only the terms linear in ϵ, we find that the generators of infinitesimal transformations for tensor reps act by touching one index at a time:

$$\begin{aligned}(T_i)^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q}, {}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} &= (T_i)^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\ &+ \delta^{a_1}_{c_1} (T_i)^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} + \ldots + \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots (T_i)^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\ &- \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} (T_i)^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} - \ldots - \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots (T_i)^{d_q}_{b_q} \,. \end{aligned} \tag{9.19}$$

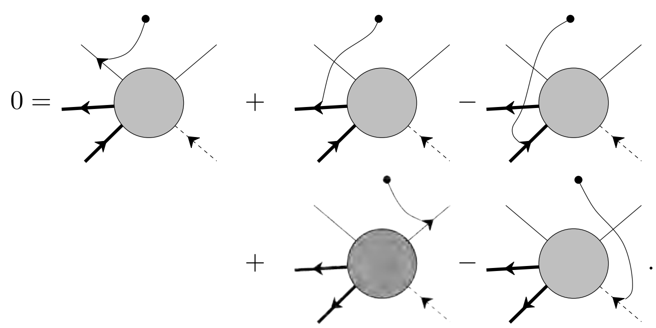
This forest of indices vanishes in the birdtrack notation, enabling us to visualize the formula for the generators of infinitesimal transformations for any tensor representation:



(9.20)

with a relative minus sign between lines flowing in opposite directions. The reader will recognize this as the Leibnitz rule.

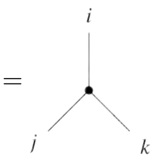
The invariance conditions take a particularly suggestive form in the birdtrack notation. Equation (9.18) amounts to the insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:

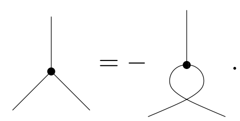


(9.21)

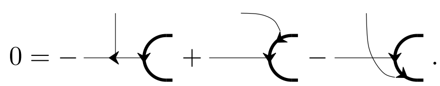
The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint rep. Where does the ugly word "adjoint" come from in this context is not obvious, but remember it this way: this is the one distinguished representation, which is intrinsic to the Lie algebra, with the explicit matrix elements $(T_i)_{jk}$ of the adjoint rep given by the fully antisymmetric structure constants iC_{ijk} of the algebra (i.e., its multiplication table under the commutator product). It's the continuous groups analogue of the multiplication table, or the regular representation for the finite groups. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex:

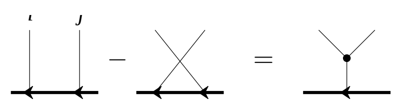
$-iC_{ijk} =$  (9.22)

 (9.23)

As all other invariant tensors, the generators must satisfy the invariance conditions (9.21):



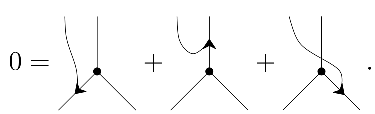
Redrawing this a little and replacing the adjoint rep generators (9.22) by the structure constants, we find that the generators obey the *Lie algebra* commutation relation

 (9.24)

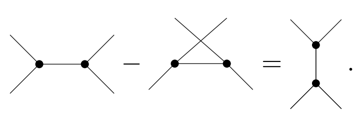
In other words, the Lie algebra commutator

$$T_iT_j \quad - \quad T_jT_i \quad = \quad iC_{ijk}T_k\,. \tag{9.25}$$

is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. Now, honestly, do you prefer the three-birdtracks equation (9.24), or the mathematician's page-long definition of the adjoint rep? It's a classic example of bad notation getting into way of understanding a relation of beautiful simplicity. The invariance condition for structure constants C_{ijk} is likewise



Rewriting this with the dot-vertex (9.22), we obtain

 (9.26)

This is the Lie algebra commutator for the adjoint rep generators, known as the *Jacobi relation* for the structure constants

$$C_{ijm}C_{mkl} - C_{ljm}C_{mki} = C_{iml}C_{jkm}\,. \tag{9.27}$$

Hence, the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.