Lie groups. Matrix representations

Reading: Chen, Ping and Wang [7] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).

Sect. 9.1 that follows is a very condensed extract of chapters 3 *Invariants and reducibility* and 4 *Diagrammatic notation* from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [9]. I am usually reluctant to use birdtrack notations in front of graduate students indoctrinated by their professors in the 1890's tensor notation, but today I'm emboldened by the very enjoyable article on *The new language of mathematics* by Dan Silver [17]. Your professor's notation is as convenient for actual calculations as -let's say- long division using roman numerals. So leave them wallowing in their early progressive rock of 1968, King Crimsons of their youth. You chill to beats younger than Windows 98, to grime, to trap, to hardvapour, to birdtracks.

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9.1 Lie groups for pedestrians

> [...] which is an expression of consecration of angular momentum.
>
> — Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G : (g,h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. The compact Lie groups that we will deploy here are a generalization of the theory of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, figure 9.1, or the manifold swept by the three Euler angles that parameterize $\mathrm{SO}(3)$ rotations.

By 'compact' one means that these parameters run over finite ranges, as opposed to parameters in hyperbolic geometries, such as Minkowsky $\mathrm{SO}(3,1)$. The groups we focus on here are compact by default, as their representations are linear, finite-dimensional matrix subgroups of the unitary matrix group $\mathrm{U}(d)$.

An element of a $[d \times d]$ -dimensional matrix representation of a *Lie group* continuously connected to identity can be written as

$$g(\boldsymbol{\phi}) = e^{i\boldsymbol{\phi}\cdot T}\,, \qquad \boldsymbol{\phi}\cdot T = \sum_{a=1}^{N} \phi_a T_a\,, \tag{9.1}$$

where $\boldsymbol{\phi}\cdot T$ is a *Lie algebra* element, T_a are matrices called 'generators', and $\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)$ are the parameters of the transformation. Repeated indices are summed throughout, and the dot product refers to a sum over Lie algebra generators. Sometimes it is convenient to use the Dirac bra-ket notation for the Euclidean product of two real vectors $x, y \in \mathbb{R}^d$, or the product of two complex vectors $x, y \in \mathbb{C}^d$, i.e., indicate complex x-transpose times y by

$$\langle x|y\rangle = x^\dagger y = \sum_{i}^{d} x_i^* y_i\,. \tag{9.2}$$

Finite unitary transformations $\exp(i\boldsymbol{\phi}\cdot T)$ are generated by sequences of infinitesimal steps of form

$$g(\boldsymbol{\delta\phi}) \simeq 1 + i\boldsymbol{\delta\phi}\cdot T\,, \quad \boldsymbol{\delta\phi} \in \mathbb{R}^N\,, \quad |\boldsymbol{\delta\phi}| \ll 1\,, \tag{9.3}$$

where T_a, the *generators* of infinitesimal transformations, are a set of linearly independent $[d \times d]$ hermitian matrices (see figure 9.2 (b)).

The reason why one can piece a global transformation from infinitesimal steps is that the choice of the "origin" in coordinatization of the group manifold sketched in figure 9.2 (a) is arbitrary. The coordinatization of the tangent space at one point on the

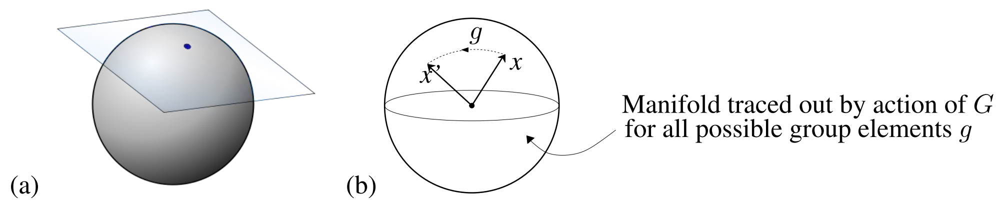

Figure 9.2: (a) Lie algebra fields $\{t_1, \cdots, t_N\}$ span the tangent space of the group orbit $\mathcal{M}_x$ at state space point x, see (9.5) (figure from WikiMedia.org). (b) A global group transformation $g : x \to x'$ can be pieced together from a series of infinitesimal steps along a continuous trajectory connecting the two points. The group orbit of state space point $x \in \mathbb{R}^d$ is the N-dimensional manifold of all actions of the elements of group G on x.

group manifold suffices to have it everywhere, by a coordinate transformation g, i.e., the new origin y is related to the old origin x by conjugation $y = g^{-1}xg$, so all tangent spaces belong the same class, they are geometrically equivalent.

Unitary and orthogonal groups are defined as groups that preserve 'length' norms, $\langle gx|gx\rangle = \langle x|x\rangle$, and infinitesimally their generators (9.3) induce no change in the norm, $\langle T_a x|x\rangle + \langle x|T_a x\rangle = 0$, hence the Lie algebra generators T_a are [3]hermitian for,

$$T_a^\dagger = T_a \,. \tag{9.4}$$

The flow field at the state space point x induced by the action of the group is given by the set of N tangent fields

$$t_a(x)_i = (T_a)_{ij} x_j \,, \tag{9.5}$$

which span the d-dimensional *group tangent space* at state space point x, parametrized by $\boldsymbol{\delta\phi}$.

For continuous groups the Lie algebra, i.e., the algebra spanned by the set of N generators T_a of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups (see figure 9.2).