## Hard work builds character

Group theory? It is all about class & character.
— Predrag Cvitanović, *One minute elevator pitch*

# 4.1 It's all about class

The essential group theory notion we shall need here is the notion of irreducible representations (irreps) and their orthogonality

**The Group Orthogonality Theorem:** Let $D_\mu$, $D_{\mu'}$ be two irreducible matrix representations of a compact group $G$ of dimensions $d_\mu$, $d_{\mu'}$, where the sum is over all elements of the group, $G = \{g\}$, and $|G|$ is their number, or the order of the group:

$$\frac{1}{|G|}\sum_{g}^{G} D^{(\mu)}(g)_a{}^b D^{(\mu')}(g^{-1})^{a'}{}_{b'} = \frac{1}{d_\mu}\delta_{\mu\mu'}\delta_a^{a'}\delta_{b'}^{b} \,.$$

This is a remarkable formula, one relation for each of the $d_\mu^2 + d_{\mu'}^2$ matrix entries. Still, the explicit matrix entries reflect largely arbitrary coordinate choices - there should a more compact statement of irreducibility, and as we shall see, there is: the "character orthogonality theorem" (4.6) that we derive next.

Show class, have pride, and display character. If you do, winning takes care of itself.
— Paul Bryant

## It takes class

In week 1 we introduced projection operators (1.33). How are they related to the character projection operators constructed in the previous lecture? While the character orthogonality might be wonderful, it is not very intuitive - it's a set of solutions to a set of symmetry-consistent orthogonality relations. You can learn a set of rules that enables you to construct a character table, but it does not tell you what it means.

In my own Group Theory book [1] I (almost) get all simple Lie algebras using projection operators constructed from invariant tensors. What that means is easier to understand for finite groups, and here I like the Harter's exposition [3] best. Harter constructs 'class operators', shows that they form a basis for the algebra of 'central' or 'all-commuting' operators, and uses their characteristic equations to construct the projection operators (1.33) from the 'structure constants' of the finite group, i.e., its class multiplication tables. Expanded, these projection operators are indeed the same as the ones obtained from character orthogonality.