**B**ASIC GROUP-THEORETIC NOTIONS are recapitulated here: groups, irreducible representations, invariants. Our notation follows birdtracks.eu.

The key result is the construction of projection operators from invariant matrices. The basic idea is simple: a hermitian matrix can be diagonalized. If this matrix is an invariant matrix, it decomposes the reps of the group into direct sums of lower-dimensional reps. Most of computations to follow implement the spectral decomposition

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_r \mathbf{P}_r \,,$$

which associates with each distinct root $\lambda_i$ of invariant matrix $\mathbf{M}$ a projection operator (A10.20):

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} \,.$$

## 1.5 Eigenvalues and eigenvectors

What is a matrix?
—Werner Heisenberg (1925)
What is the matrix?
—-Keanu Reeves (1999)

**Eigenvalues** of a $[d \times d]$ matrix $\mathbf{M}$ are the roots of its characteristic polynomial

$$\det(\mathbf{M} - \lambda \mathbf{1}) = \prod (\lambda_i - \lambda) = 0 \,. \tag{1.27}$$

Given a nonsingular matrix $\mathbf{M}$, with all $\lambda_i \neq 0$, acting on $d$-dimensional vectors $\mathbf{x}$, we would like to determine *eigenvectors* $\mathbf{e}^{(i)}$ of $\mathbf{M}$ on which $\mathbf{M}$ acts by scalar multiplication by eigenvalue $\lambda_i$

$$\mathbf{M}\,\mathbf{e}^{(i)} = \lambda_i \mathbf{e}^{(i)} \,. \tag{1.28}$$

If $\lambda_i \neq \lambda_j$, $\mathbf{e}^{(i)}$ and $\mathbf{e}^{(j)}$ are linearly independent. There are at most $d$ distinct eigenvalues which we order by their real parts, $\mathrm{Re}\,\lambda_i \geq \mathrm{Re}\,\lambda_{i+1}$.

If all eigenvalues are distinct, $\mathbf{e}^{(j)}$ are $d$ linearly independent vectors which can be used as a (non-orthogonal) basis for any $d$-dimensional vector $\mathbf{x} \in \mathbb{R}^d$

$$\mathbf{x} = x_1\,\mathbf{e}^{(1)} + x_2\,\mathbf{e}^{(2)} + \cdots + x_d\,\mathbf{e}^{(d)} \,. \tag{1.29}$$

However, $r$, the number of distinct eigenvalues, is in general smaller than the dimension of the matrix, $r \leq d$ (see example 1.3).

From (1.28) it follows that

$$(\mathbf{M} - \lambda_i \mathbf{1})\,\mathbf{e}^{(j)} = (\lambda_j - \lambda_i)\,\mathbf{e}^{(j)} \,,$$

matrix $(\mathbf{M} - \lambda_i \mathbf{1})$ annihilates $\mathbf{e}^{(i)}$, thus the product of all such factors annihilates any vector, and the matrix $\mathbf{M}$ satisfies its characteristic equation

$$\prod_{i=1}^{d} (\mathbf{M} - \lambda_i \mathbf{1}) = 0\,. \tag{1.30}$$

This humble fact has a name: the Hamilton-Cayley theorem. If we delete one term from this product, we find that the remainder projects $\mathbf{x}$ from (1.29) onto the corresponding eigenspace:

$$\prod_{j \neq i} (\mathbf{M} - \lambda_j \mathbf{1})\mathbf{x} = \prod_{j \neq i} (\lambda_i - \lambda_j)\, x_i \mathbf{e}^{(i)}\,.$$

Dividing through by the $(\lambda_i - \lambda_j)$ factors yields the *projection operators*

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j}\,, \tag{1.31}$$

which are *orthogonal* and *complete*:

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_j\,, \quad (\text{no sum on } j)\,, \qquad \sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1}\,, \tag{1.32}$$

with the dimension of the $i$th subspace given by $d_i = \mathrm{tr}\, \mathbf{P}_i$ . For each distinct eigenvalue $\lambda_i$ of $\mathbf{M}$,

$$(\mathbf{M} - \lambda_j \mathbf{1})\mathbf{P}_j = \mathbf{P}_j (\mathbf{M} - \lambda_j \mathbf{1}) = 0\,, \tag{1.33}$$

the colums/rows of $\mathbf{P}_j$ are the right/left eigenvectors $\mathbf{e}^{(j)}$, $\mathbf{e}_{(j)}$ of $\mathbf{M}$ which (provided $\mathbf{M}$ is not of Jordan type) span the corresponding linearized subspace.

The main take-home is that once the distinct non-zero eigenvalues $\{\lambda_i\}$ are computed, projection operators are polynomials in $\mathbf{M}$ which need no further diagonalizations or orthogonalizations.

It follows from the characteristic equation (1.33) that $\lambda_i$ is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\,\mathbf{P}_i = \lambda_i \mathbf{P}_i \qquad (\text{no sum on } i)\,. \tag{1.34}$$

Using $\mathbf{M} = \mathbf{M}\,\mathbf{1}$ and completeness relation (1.32) we can rewrite $\mathbf{M}$ as

$$\mathbf{M} = \lambda_1 \mathbf{P}_1 + \lambda_2 \mathbf{P}_2 + \cdots + \lambda_d \mathbf{P}_d\,. \tag{1.35}$$

Any matrix function $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace, $f(\mathbf{M})\,\mathbf{P}_i = f(\lambda_i)\,\mathbf{P}_i$ , and is thus easily evaluated through its *spectral decomposition*

$$f(\mathbf{M}) = \sum_i f(\lambda_i) \mathbf{P}_i\,. \tag{1.36}$$

This, of course, is the reason why anyone but a fool works with irreducible reps: they reduce matrix (AKA "operator") evaluations to manipulations with numbers.

By (1.33) every column of $\mathbf{P}_i$ is proportional to a right eigenvector $\mathbf{e}^{(i)}$, and its every row to a left eigenvector $\mathbf{e}_{(i)}$. In general, neither set is orthogonal, but by the idempotence condition (1.32), they are mutually orthogonal,

$$\mathbf{e}_{(i)} \cdot \mathbf{e}^{(j)} = c_j \, \delta_i^j \,. \tag{1.37}$$

The non-zero constant $c_j$ is convention dependent and not worth fixing, unless you feel nostalgic about Clebsch-Gordan coefficients. We shall set $c_j = 1$. Then it is convenient to collect all left and right eigenvectors into a single matrix.

## A10.2 Invariants and reducibility

What follows is a bit dry, so we start with a motivational quote from Hermann Weyl on the "so-called first main theorem of invariant theory": [5]

"*All invariants are expressible in terms of a finite number among them*. We cannot claim its validity for every group *G*; rather, it will be our chief task to investigate for each particular group whether a finite integrity basis exists or not; the answer, to be sure, will turn out affirmative in the most important cases."

It is easy to show that any rep of a finite group can be brought to unitary form, and the same is true of all compact Lie groups. Hence, in what follows, we specialize to unitary and hermitian matrices.

### A10.2.1 Projection operators

For $\mathbf{M}$ a hermitian matrix, there exists a diagonalizing unitary matrix $\mathbf{C}$ such that

$$\mathbf{CMC}^{\dagger} = \left[\begin{array}{ccc|cccc|cc} \lambda_1 & \dots & 0 & & & & & & \\ & \ddots & & & & 0 & & & 0 \\ 0 & \dots & \lambda_1 & & & & & & \\ \hline & & & \lambda_2 & 0 & \dots & 0 & & \\ & & & 0 & \lambda_2 & & & & \\ & 0 & & \vdots & & \ddots & \vdots & & 0 \\ & & & 0 & & \dots & \lambda_2 & & \\ \hline & & & & & & & \lambda_3 & \dots \\ & 0 & & & & 0 & & \vdots & \ddots \end{array}\right] . \tag{A10.18}$$

Here $\lambda_i \neq \lambda_j$ are the $r$ distinct roots of the minimal *characteristic* (or *secular*) polynomial

$$\prod_{i=1}^{r} (\mathbf{M} - \lambda_i \mathbf{1}) = 0 . \tag{A10.19}$$

In the matrix $\mathbf{C}(\mathbf{M} - \lambda_2 \mathbf{1})\mathbf{C}^{\dagger}$ the eigenvalues corresponding to $\lambda_2$ are replaced by zeroes:

$$\left[\begin{array}{cc|ccc|ccc} \lambda_1 - \lambda_2 & & & & & & & \\ & \lambda_1 - \lambda_2 & & & & & & \\ \hline & & 0 & & & & & \\ & & & \ddots & & & & \\ & & & & 0 & & & \\ \hline & & & & & \lambda_3 - \lambda_2 & & \\ & & & & & & \lambda_3 - \lambda_2 & \\ & & & & & & & \ddots \end{array}\right] ,$$

and so on, so the product over all factors $(\mathbf{M} - \lambda_2 \mathbf{1})(\mathbf{M} - \lambda_3 \mathbf{1}) \dots$, with exception of the $(\mathbf{M} - \lambda_1 \mathbf{1})$ factor, has nonzero entries only in the subspace associated with $\lambda_1$:

$$\mathbf{C} \prod_{j \neq 1} (\mathbf{M} - \lambda_j \mathbf{1}) \mathbf{C}^{\dagger} = \prod_{j \neq 1} (\lambda_1 - \lambda_j) \left[\begin{array}{ccc|cccc} 1 & 0 & 0 & & & & \\ 0 & 1 & 0 & & 0 & & \\ 0 & 0 & 1 & & & & \\ \hline & & & 0 & & & \\ & & & & 0 & & \\ & 0 & & & & 0 & \\ & & & & & & \ddots \end{array}\right] .$$

Thus we can associate with each distinct root $\lambda_i$ a *projection operator* $\mathbf{P}_i$,

$$\mathbf{P}_i = \prod_{j \neq i} \frac{\mathbf{M} - \lambda_j \mathbf{1}}{\lambda_i - \lambda_j} , \tag{A10.20}$$

which acts as identity on the $i$th subspace, and zero elsewhere. For example, the projection operator onto the $\lambda_1$ subspace is

$$\mathbf{P}_1 = \mathbf{C}^\dagger \left[ \begin{array}{ccc|cccc} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ \hline & & & 0 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{array} \right] \mathbf{C} \,. \tag{A10.21}$$

The diagonalization matrix $\mathbf{C}$ is deployed in the above only as a pedagogical device. The whole point of the projector operator formalism is that we *never* need to carry such explicit diagonalization; all we need are whatever invariant matrices $\mathbf{M}$ we find convenient, the algebraic relations they satisfy, and orthonormality and completeness of $\mathbf{P}_i$: The matrices $\mathbf{P}_i$ are *orthogonal*

$$\mathbf{P}_i \mathbf{P}_j = \delta_{ij} \mathbf{P}_j \,, \qquad (\text{no sum on } j) \,, \tag{A10.22}$$

and satisfy the *completeness relation*

$$\sum_{i=1}^{r} \mathbf{P}_i = \mathbf{1} \,. \tag{A10.23}$$

As $\operatorname{tr}(\mathbf{C}\mathbf{P}_i\mathbf{C}^\dagger) = \operatorname{tr}\mathbf{P}_i$, the dimension of the $i$th subspace is given by

$$d_i = \operatorname{tr}\mathbf{P}_i \,. \tag{A10.24}$$

It follows from the characteristic equation (A10.19) and the form of the projection operator (A10.20) that $\lambda_i$ is the eigenvalue of $\mathbf{M}$ on $\mathbf{P}_i$ subspace:

$$\mathbf{M}\mathbf{P}_i = \lambda_i \mathbf{P}_i \,, \qquad (\text{no sum on } i) \,. \tag{A10.25}$$

Hence, any matrix polynomial $f(\mathbf{M})$ takes the scalar value $f(\lambda_i)$ on the $\mathbf{P}_i$ subspace

$$f(\mathbf{M})\mathbf{P}_i = f(\lambda_i)\mathbf{P}_i \,. \tag{A10.26}$$

This, of course, is the reason why one wants to work with irreducible reps: they reduce matrices and "operators" to pure numbers.

### A10.2.2 Irreducible representations

Suppose there exist several linearly independent invariant $[d\times d]$ hermitian matrices $\mathbf{M}_1, \mathbf{M}_2, \ldots$, and that we have used $\mathbf{M}_1$ to decompose the $d$-dimensional vector space $V = V_1 \oplus V_2 \oplus \cdots$. Can $\mathbf{M}_2, \mathbf{M}_3, \ldots$ be used to further decompose $V_i$? Further decomposition is possible if, and only if, the invariant matrices commute:

$$[\mathbf{M}_1, \mathbf{M}_2] = 0 \,, \tag{A10.27}$$

or, equivalently, if projection operators $\mathbf{P}_j$ constructed from $\mathbf{M}_2$ commute with projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$,

$$\mathbf{P}_i\mathbf{P}_j = \mathbf{P}_j\mathbf{P}_i \,. \tag{A10.28}$$

Usually the simplest choices of independent invariant matrices do not commute. In that case, the projection operators $\mathbf{P}_i$ constructed from $\mathbf{M}_1$ can be used to project commuting pieces of $\mathbf{M}_2$:

$$\mathbf{M}_2^{(i)} = \mathbf{P}_i\mathbf{M}_2\mathbf{P}_i \,, \qquad \text{(no sum on } i) \,.$$

That $\mathbf{M}_2^{(i)}$ commutes with $\mathbf{M}_1$ follows from the orthogonality of $\mathbf{P}_i$:

$$[\mathbf{M}_2^{(i)}, \mathbf{M}_1] = \sum_j \lambda_j [\mathbf{M}_2^{(i)}, \mathbf{P}_j] = 0 \,. \tag{A10.29}$$

Now the characteristic equation for $\mathbf{M}_2^{(i)}$ (if nontrivial) can be used to decompose $V_i$ subspace.

An invariant matrix $\mathbf{M}$ induces a decomposition only if its diagonalized form (A10.18) has more than one distinct eigenvalue; otherwise it is proportional to the unit matrix and commutes trivially with all group elements. A rep is said to be *irreducible* if all invariant matrices that can be constructed are proportional to the unit matrix.