Example 2.5. ***Projection operators for cyclic group C_N.*** 2CB

Consider a cyclic group $C_N = \{e, g, g^2, \cdots g^{N-1}\}$, and let $M = D(g)$ be a $[2N\times 2N]$ representation of the one-step shift g. In the projection operator formulation (1.31), the N distinct eigenvalues of M, the Nth roots of unity $\lambda_n = \lambda^n$, $\lambda = \exp(i\,2\pi/N)$, $n = 0, \ldots N - 1$, split the $2N$-dimensional space into N 2-dimensional subspaces by means of projection operators

$$P_n = \prod_{m\neq n} \frac{M - \lambda_m I}{\lambda_n - \lambda_m} = \prod_{m=1}^{N-1} \frac{\lambda^{-n} M - \lambda^m I}{1 - \lambda^m}\,, \tag{2.7}$$

where we have multiplied all denominators and numerators by λ^{-n}. The numerator is now a matrix polynomial of form $(x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$, with the zeroth root $(x-\lambda^0) = (x-1)$ quotiented out from the defining matrix equation $M^N - 1 = 0$. Using

$$\frac{1-x^N}{1-x} = 1 + x + \cdots + x^{N-1} = (x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$$

we obtain the projection operator in form of a discrete Fourier sum (rather than the product (1.31)),

$$P_n = \frac{1}{N}\sum_{m=0}^{N-1} e^{i\,\frac{2\pi}{N} nm}\, M^m\,.$$

This form of the projection operator is the simplest example of the key group theory tool, projection operator expressed as a sum over characters,

$$P_n = \frac{1}{|G|}\sum_{g\in G} \bar{\chi}(g) D(g)\,,$$

upon which stands all that follows in this course. *(B. Gutkin and P. Cvitanović)*