Exercises

16.1. **Gravity tensors.** In this problem we will apply diagrammatic methods ("birdtracks") to construct and count the numbers of independent components of the "irreducible rank-four gravity curvature tensors." However, any notation that works for you is OK, as long as you obtain the same irreps and their dimensions. The goal of this exercise (longish, as much of it is the recapitulation of the material covered in the book) is to give you basic understanding for how Young tableaux work for groups other than U(n). We start with

Part 1 : **U(n) Young tableaux decomposition.**

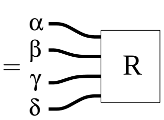
(a) The Riemann-Christoffel curvature tensor of general relativity has the following symmetries (see, for example, Weinberg [10] or the Riemann curvature tensor wiki):

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{16.1}$$

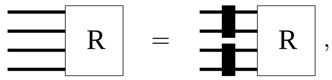
$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{16.2}$$

$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0\,. \tag{16.3}$$

Introducing a birdtrack notation for the Riemann tensor

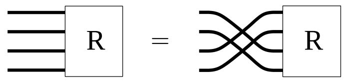
$R_{\alpha\beta\gamma\delta} =$  , (16.4) check that we can state the above symmetries as

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$



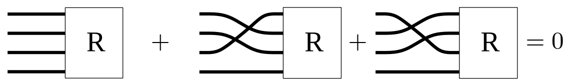
(16.5)

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$



(16.6)

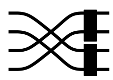
$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0$$

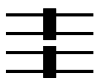


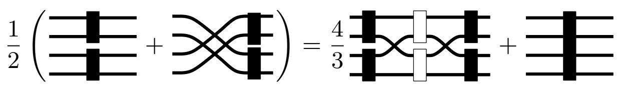
(16.7)

The first condition says that R lies in the ⊗ subspace.

(b) The second condition says that R lies in the ↔ interchange-symmetric subspace.

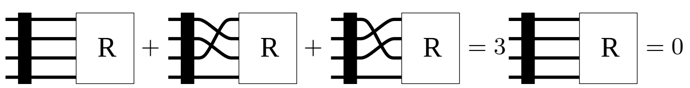
Use the characteristic equation for 

to split  into the and irreps:



(16.8)

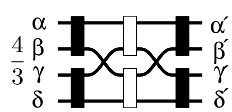
(c) Show that the third condition (16.7) says that R has no components in the irrep:



(16.9)

Hence, the symmetries of the Riemann tensor are summarized by the irrep projection operator [8]:

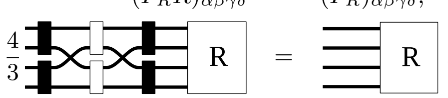
$$(P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} =$$



(16.10)

(d) Verify that the Riemann tensor is in the subspace

$$(P_R R)_{\alpha\beta\gamma\delta} = (P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} R_{\alpha'\beta'\gamma'\delta'} = R_{\alpha\beta\gamma\delta}$$



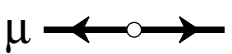
(16.11)

(e) Compute the number of independent components of the Riemann tensor $\mathbf{R}_{\alpha\beta\gamma\delta}$ by taking the trace of the irrep projection operator:

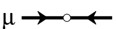
$$d_R = \operatorname{tr} \mathbf{P}_R = \frac{n^2(n^2-1)}{12} \,. \tag{16.12}$$

Part 2 : **SO(n) Young tableaux decomposition**

The Riemann tensor has the symmetries of the $\boxplus$ irrep of U(n). However, gravity is also characterized by the symmetric tensor $g_{\alpha\beta}$, that reduces the symmetry to a local SO(n) invariance (more precisely SO($1, n-1$), but compactness is not important here). The extra invariants built from $g_{\alpha\beta}$'s decompose U(n) reps into sums of SO(n) reps. Orthogonal group SO(n) is the group of transformations that leaves invariant a symmetric quadratic form $(q, q) = g_{\mu\nu} q^\mu q^\nu$, with a primitive invariant rank-2 tensor:

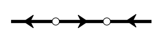
$$g_{\mu\nu} = g_{\nu\mu} = \mu$$  $$\nu \qquad \mu, \nu = 1, 2, \ldots, n\,. \qquad (16.13)$$

If (q, q) is an invariant, so is its complex conjugate $(q, q)^* = g^{\mu\nu} q_\mu q_\nu$, and

$$g^{\mu\nu} = g^{\nu\mu} = \mu$$  $$\nu \qquad (16.14)$$

is also an invariant tensor. The matrix $A_\mu^\nu = g_{\mu\sigma} g^{\sigma\nu}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional rep. A convenient normalization is

$$g_{\mu\sigma} g^{\sigma\nu} = \delta_\mu^\nu$$

 $=$. (16.15)

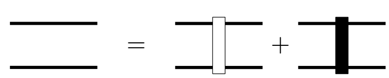
As the indices can be raised and lowered at will, nothing is gained by keeping the arrows. Our convention will be to perform all contractions with metric tensors with upper indices and omit the arrows and the open dots:

$$g^{\mu\nu} \equiv \mu$$ $$\nu\,. \qquad (16.16)$$

The U(n) 2-index tensors can be decomposed into a sum of their symmetric and antisymmetric parts. Specializing to the subgroup SO(n), the rule is to lower all indices on all tensors, and the symmetrization projection operator is written as

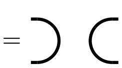
$$\begin{aligned} S_{\mu\nu,\rho\sigma} &= g_{\rho\rho'} g_{\sigma\sigma'} S_{\mu\nu,}{}^{\rho'\sigma'} \\ &= \frac{1}{2}\left(g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma}\right) \end{aligned}$$

From now on, we drop all arrows and $g^{\mu\nu}$'s and write the decomposition into symmetric and antisymmetric parts as

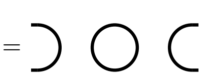


$$g_{\mu\sigma} g_{\nu\rho} = \frac{1}{2}(g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma}) + \frac{1}{2}(g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma})\,. \qquad (16.17)$$

The new invariant tensor, specific to SO(n), is the index contraction:

$$\mathbf{T}_{\mu\nu,\rho\sigma} = g_{\mu\nu} g_{\rho\sigma}\,, \qquad \mathbf{T} =$$  . (16.18)

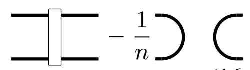
Its characteristic equation

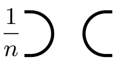
$$\mathbf{T}^2 =$$  $$= n\mathbf{T} \qquad (16.19)$$

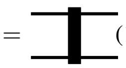
yields the trace and the traceless part projection operators. As $\mathbf{T}$ is symmetric, $S\mathbf{T} = \mathbf{T}$, only the symmetric subspace is reduced by this invariant.

(f) Show that SO(n) 2-index tensors decompose into three irreps:

traceless symmetric:

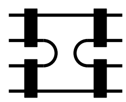
$$(P_2)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right) - \frac{1}{n}g_{\mu\nu}g_{\rho\sigma} =$$  (16.20)

singlet: $$(P_1)_{\mu\nu,\rho\sigma} = \frac{1}{n}g_{\mu\nu}g_{\rho\sigma} = \frac{1}{n}$$  , (16.21)

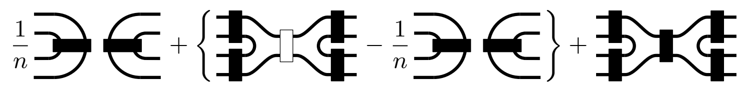
antisymmetric: $$(P_3)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right) =$$  (16.22)

What are the dimensions of the three irreps?

(g) In the same spirit, the U(n) irrep  is decomposed by the SO(n) intermediate 2-index state invariant matrix

$$\mathbf{Q} =$$  . (16.23)

Show that the intermediate 2-index subspace splits into three irreducible reps by (16.20) – (16.22):

$$\mathbf{Q} =$$ 

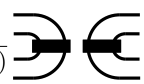
$$= \mathbf{Q}_0 + \mathbf{Q}_S + \mathbf{Q}_A \,. \qquad (16.24)$$

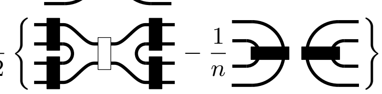
Show that the antisymmetric 2-index state does not contribute

$$\mathbf{P_R}\mathbf{Q}_A = 0\,. \qquad (16.25)$$

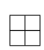
(Hint: The Riemann tensor is symmetric under the interchange of index pairs.)

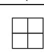
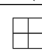
(h) Fix the normalization of the remaining two projection operators by computing $\mathbf{Q}_S^2, \mathbf{Q}_0^2$:

$$\mathbf{P}_0 = \frac{2}{n(n-1)}$$  , (16.26)

$$\mathbf{P}_S = \frac{4}{n-2}$$  (16.27)

and compute their dimensions.

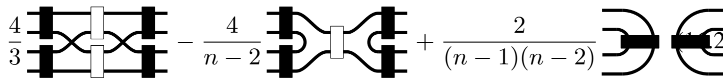
This completes the SO(n) reduction of the  U(n) irrep (16.11):

U(n)	$\to$	SO(n)				
	$\to$		+		+	$\circ$
$\mathbf{P}_R$	=	$\mathbf{P}_W$	+	$\mathbf{P}_S$	+	$\mathbf{P}_0$
$\frac{n^2(n^2-1)}{12}$	=	$\frac{(n+2)(n+1)n(n-3)}{12}$	+	$\frac{(n+2)(n-1)}{2}$	+	1

(16.28)

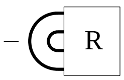
The projection operator for the SO(n) traceless  irrep is:

$$\mathbf{P}_W = \mathbf{P}_R - \mathbf{P}_S - \mathbf{P}_0$$

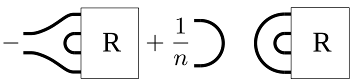
$\mathbf{P}_W =$  (16.29)

(i) The above three projection operators project out the standard, SO(n)-irreducible general relativity tensors:

Curvature scalar:

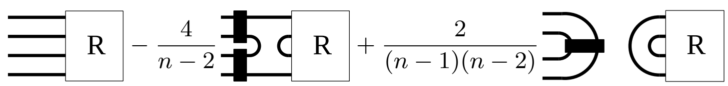
$R = -$  $= R^{\mu}{}_{\nu\mu}{}^{\nu}$ (16.30)

Traceless Ricci tensor:

$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = -$  (16.31)

Weyl tensor:

$$C_{\lambda\mu\nu\kappa} = (\mathbf{P}_W R)_{\lambda\mu\nu\kappa}$$

$=$ 

$$= R_{\lambda\mu\nu\kappa} + \frac{1}{n-2}\left(g_{\mu\nu}R_{\lambda\kappa} - g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\kappa}R_{\lambda\nu} + g_{\lambda\kappa}R_{\mu\nu}\right) - \frac{1}{(n-1)(n-2)}(g_{\lambda\kappa}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\kappa})R\,. \qquad (16.32)$$

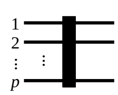
The numbers of independent components of these tensors are given by the dimensions of corresponding irreducible subspaces in (16.28).

What is the lowest dimension in which the Ricci tensor contributes? the Weyl tensor contributes? Show that in 2, respectively 3 dimensions, we have

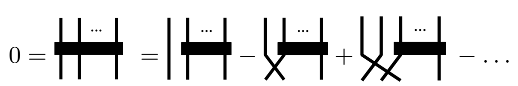
$$\begin{aligned} n=2: \quad R_{\lambda\mu\nu\kappa} &= (P_0 R)_{\lambda\mu\nu\kappa} = \tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R \\ n=3: \quad &= g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu} - g_{\lambda\kappa}R_{\mu\nu} \\ &\quad -\tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R\;. \end{aligned} \qquad (16.33)$$

(j) The last example of this exercise is an application of birdtracks to general relativity index manipulations. The object is to find the characteristic equation for the Riemann tensor in *four dimensions*.

The antisymmetrization tensor $A_{a_1 a_2 \ldots,}{}^{b_p \ldots b_2 b_1}$ has nonvanishing components, only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no nonvanishing components:

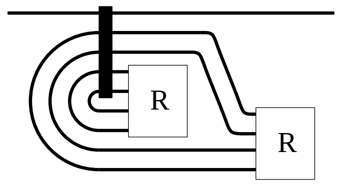
 $= 0 \qquad \text{if } p > n\,.$ (16.34)

This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n+1$ case

 (16.35)

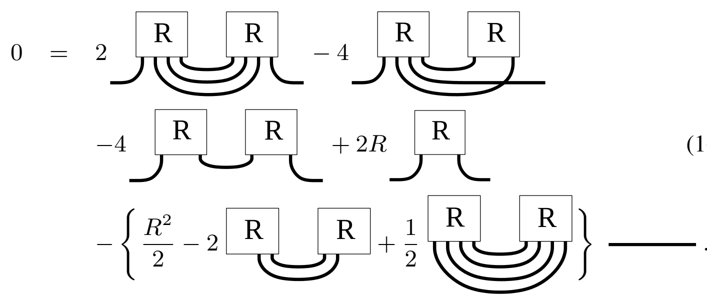
Contract (16.34) with two Riemann tensors:

$$0 =$$



, (16.36)

and obtain the characteristic equation by expanding with (16.35):



(16.37)