An overview of the course

If I had had more time, I would have written less
— Blaise Pascal, a remark made to a correspondent

This whole course has only one message:

If you have a symmetry, **use it!**

If you are reading this, think of it as an opportunity to rethink the key ideas of this branch of mathematics, take with you the few essential insights that may serve you well in your career later on.

Linear algebra

Projection operators (1.33): eigenvalues of a matrix split a vector space into subspaces.

Finite groups

Groups, permutations, group multiplication tables, rearrangement theorem, subgroups, cosets, classes.

Representation theory

Irreps, regular representation. So far, everything was intuitive: a representation of a group was bunch of 0's and 1's indicating how a group operation permutes physical objects. But now the first surprise:

Any representation of any finite group can be put into unitary form, and so complex-valued vector spaces and unitary representation matrices make their entrance.

Characters

Schur's Lemma. Unitary matrices can be diagonalized, and from that follows the Wonderful Orthogonality Theorem for Characters (coordinate independent, intrinsic numbers), and the full reducibility of any representation of any finite group.

Classes

The algebra of central or 'all-commuting' class operators, connects the reduction in terms of characters to the projection operators of week 1. The key idea:

Define a group by what objects (primitive invariant tensors) it leaves invariant.

Continuous groups

Lie groups. Matrix representations. Invariant tensors. Lie algebra. Adjoint representation, Jacobi relation. Birdtracks.

SO(3) characters; O(2) symmetry sliced

(a) Group integrals. SO(3) character orthogonality.

Simple Lie algebras; SU(3)

The next profound shift:

So far all our group notions were based on tangible, spatial intuition: permutations, reflections, rotations. But now Lie groups take on a life of their own.

(a) The SO(3) theory of angular momenta generalizes to Killing-Cartan lattices, and a fully abstract enumeration of all possible semi-simple compact Lie groups.

(b) SU(2) is promoted to an *internal* isospin symmetry, decoupled from our Euclidean spatial intuition. Modern particle physics is born, with larger and larger internal symmetry groups, tacked onto higher and higher dimensional continuum spacetimes.

Young tableaux

We have come full circle now: as a much simpler alternative to the Cartan-Killing construction, irreps of the *finite* symmetric group S_n classify the irreps of the *continuous* SU(n) symmetry multi-particle states.

Wigner 3- and 6-j coefficients (not included in the final)

The goal of group theory is to predict measurable numbers, numbers independent of any particular choice of coordinate. The full reducibility says that any such number is built from 3- and 6-j coefficients: they are the total content of group theory.