permutations

Predrag Cvitanović
Judith M. Alcock-Zeilinger
Stefan Keppeler

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Plan

- construct bases for color space
- using projectors onto $\mathrm{SU}(N)$ irreps
- written as birdtracks

definition : finite group G

consists of a set of $|G|$ elements

$$G = \{e, g_2, \dots, g_{|G|}\}$$

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3. identity e: $ge = eg = g$ for all $g \in G$
4. inverse g^{-1}: for every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that $hg = gh = e$.

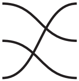
order of the group = number of elements $|G|$

example : permutation group

three standard notations for permutations

$$\rho \;=\; \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$

$$\pi \;=\; \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}}_{\textbf{two-line notation}} \;=\; \underbrace{(132)}_{\textbf{cycle notation}} \;=$$



,

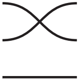
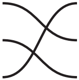
birdtrack notation

all mean : permute objects $1, 2, 3$, so that after the permutation

$$\pi(1) = 3\,,\;\; \pi(2) = 1\,,\;\; \pi(3) = 2$$

permutation multiplication

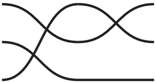
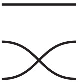
to compose permutations

$\rho = (12) =$  , followed by $\pi = (132) =$ 

multiply cycles

$$\pi \circ \rho = \underbrace{(132)(12)}_{\text{omit '}\circ\text{'}} = (1)(23) = \underbrace{(23)}_{\text{omit one-cycles}}$$

or compose diagrams

$\pi \circ \rho =$  $=$ 

S_n : permutation group on n quarks

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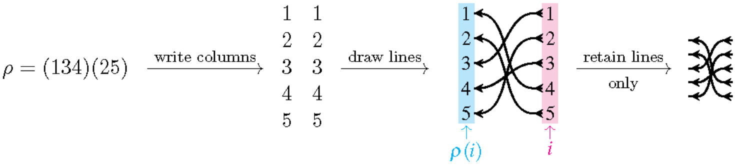

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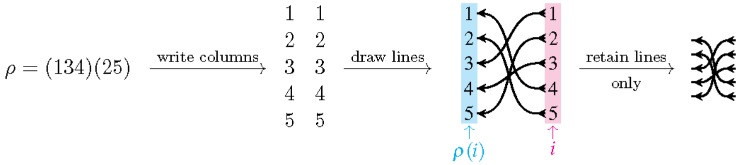
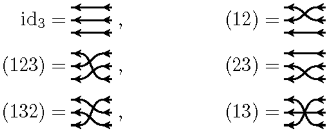
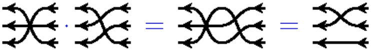


- permutation group S_3 elements



- birdtrack multiplication :
  connect lines and straighten them out



- birdtrack inverse

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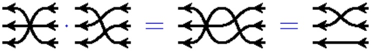

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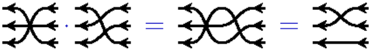

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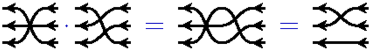

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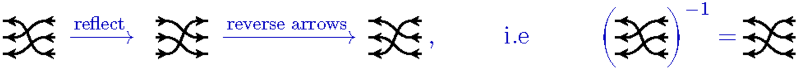

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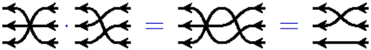

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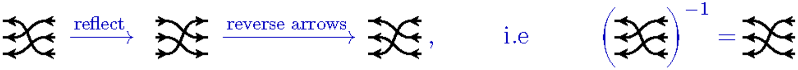
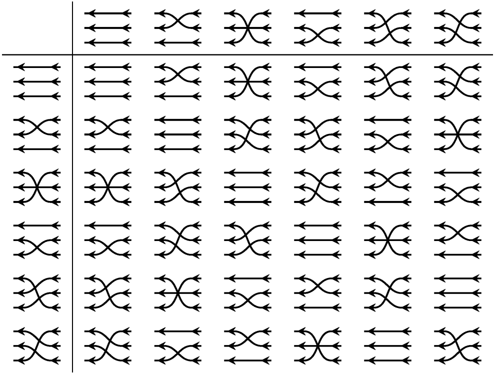


closure, associativity, identity, inverse : it's a group !

S_3 multiplication table



definition : algebra over group elements

the vector space $\mathcal{A} = \mathbb{C}[G]$ constructed from linear combinations of group elements

$$a = \sum_g \lambda_g g \,, \qquad g \in G \,, \quad \lambda_g \in \mathbb{C}$$

for example :

$=$  -5  $+2i$  , $\qquad a \in \mathbb{C}[\mathrm{S}_4]$

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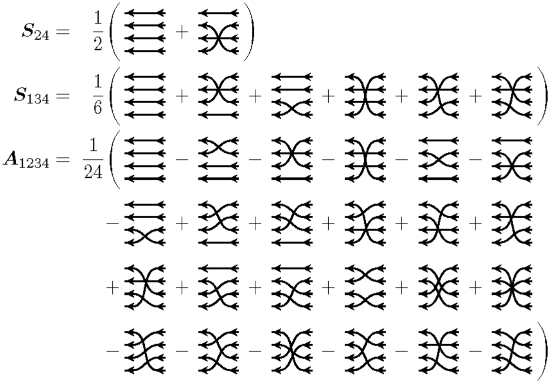
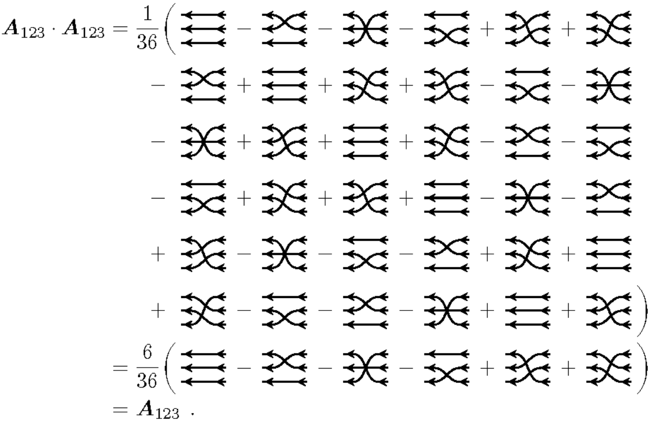
is an *algebra*, with $\mathcal{A} \cdot \mathcal{A} \to \mathcal{A}$ distributive multiplication

$$(a+b)\,c = ac + bc \,, \qquad a, b, c \in \mathcal{A}$$

examples of algebra elements



birdtrack multiplication in long hand



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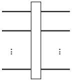
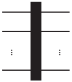
compact birdtrack notation : (anti)symmetrizers

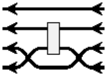
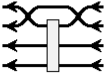
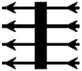
denote symmetrizers S / anti-symmetrizers A

$$S = \frac{1}{n!} \sum_{\pi \in S_n} \pi \qquad \text{and} \qquad A = \frac{1}{n!} \sum_{\pi \in S_n} \operatorname{sign}(\pi)\, \pi$$

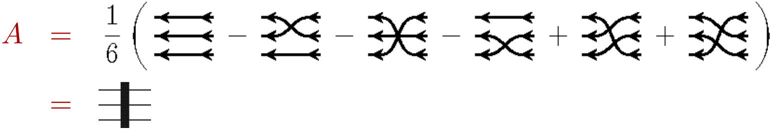
by white, black bars

$S =$  and $A =$  .

compact birdtrack notation for algebra elements

$S_{24} =$ , $S_{134} =$ , $A_{1234} =$ 

birdtrack computations are compact



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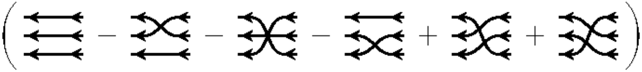
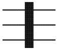
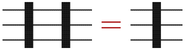
$A = \frac{1}{6}$ 

$=$ 

antisymmetrize twice :

$A^2 =$ 

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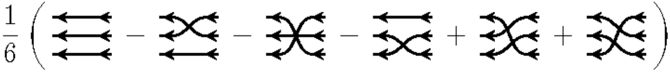
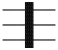
$A =$ 

$=$ 

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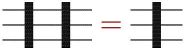
$A^2 =$ 

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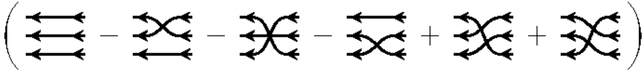
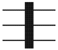
$A = \frac{1}{6}$ 

$=$ 

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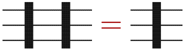
$A^2 =$ 

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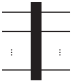
$A^2 = A$, *i.e.* S, A are *projection operators*.
Operating on what ?

what does operator A do ?

In the full index notation

$$A_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} - \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$

$$A =$$ 

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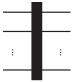
$A =$ 

is a (tensorial) matrix that acts on p copies of vector space V

$$A : V^p \to V^p$$

practice : birdtracks $\leftrightarrow$ indices

operation of permuting tensor indices is a linear operation, represented by a $[d \times d]$ matrix:

$$\sigma_\alpha{}^\beta = \sigma_{b_1 \dots b_p}{}^{a_1 a_2 \dots a_q}{}_{,}{}^{d_p \dots d_1}_{c_q \dots c_2 c_1} \,.$$

for 2-index tensors, there are two permutations:

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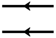
identity: $\mathbf{1}_{ab}{}^{cd} = \delta_a^d \delta_b^c =$ 

flip: $\sigma_{(12)ab}{}^{cd} = \delta_a^c \delta_b^d =$  .

practice : birdtracks $\leftrightarrow$ indices

For 3-index tensors, there are six permutations:

$$\mathbf{1}_{a_1a_2a_3}{}^{b_3b_2b_1} = \delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\delta_{a_3}^{b_3} =$$

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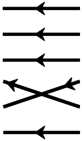

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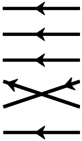


WOA!

- (conventional) Young operators are not Hermitian
  $\rightsquigarrow$ do not yield orthogonal bases

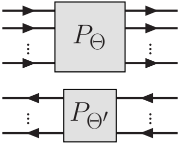
$Y_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \frac{4}{3}$  , $\quad Y_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = \frac{4}{3}$ 

- Hermitian Young operators ☺

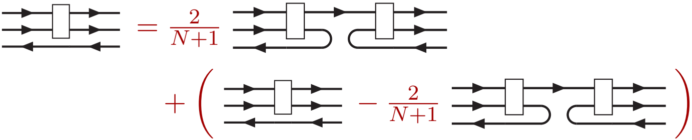
$P_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \frac{4}{3}$  , $\quad P_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = \frac{4}{3}$ 

SK & M. Sjödahl, J. Math. Phys. **55** (2014) 021702, `arXiv:1307.6147`
J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051702, `arXiv:1610.10088`
J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051703, `arXiv:1610.08802`

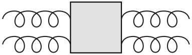
- apply two Hermitian Young operators



- further decompose by subtracting contractions, e.g.



	number of multiplets		dimension of color space	
	$N = 3$	$N = \infty$	$N = 3$	$N = \infty$
$A^2 \to A^2$	6	7	8	9
$A^3 \to A^3$	29	51	145	265
$A^4 \to A^4$	166	513	3 598	14 833
$A^5 \to A^5$	1 002	6 345	107 160	1 334 961



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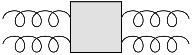

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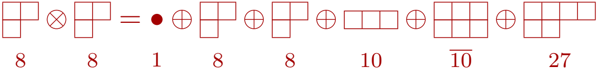

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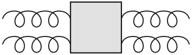

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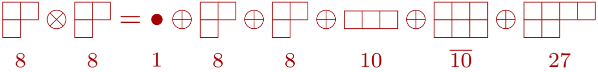
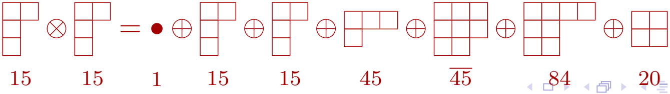
$A^4 \to A^4$	166	513	3 598	14 833
$A^5 \to A^5$	1 002	6 345	107 160	1 334 961



$N=3$



$N=4$

constructing multiplet bases

- quarks only ⇝ Hermitian Young operators[*,†,‡]
- gluons only[§]
- quarks & anti-quarks
  work in progress (Keppeler & Alcock-Zeilinger)
- quarks, anti-quarks & gluons ⇝ (at least) two strategies[¶]
  work in progress (Alcock-Zeilinger & Keppeler & Sjödahl & Thorén)

working with multiplet bases

- download bases for 6 external partons[¶]
- software
  M. Sjödahl: ColorMath/ColorFull (Mathematica/C++ packages)
- multiplet bases ⇝ Wigner 3j & 6j coefficients[¶]

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[*] S. Keppeler and M. Sjödahl, J. Math. Phys. **55**, 021702 (2014).

[†] J. Alcock-Zeilinger and H. Weigert, J. Math. Phys. **58**, 051702 (2017).

[‡] J. Alcock-Zeilinger and H. Weigert, J. Math. Phys. **58**, 051703 (2016).

[§] S. Keppeler and M. Sjödahl, J. High Energy Phys. **2012**, 1–49 (2012).

[¶] M. Sjödahl and J. Thorén, J. High Energy Phys. **09**, 55 (2015).