

permutations

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Plan

- ▶ construct bases for color space
- ▶ using projectors onto $SU(N)$ irreps
- ▶ written as birdtracks



definition : finite group G

consists of a set of $|G|$ elements



$$G = \{e, g_2, \dots, g_{|G|}\}$$

and a group multiplication rule $g_j g_i$ with



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3. identity e : $ge = eg = g$ for all $g \in G$
4. inverse g^{-1} : for every $g \in G$, there exists a unique element $h = g^{-1} \in G$ such that $hg = gh = e$.

order of the group = number of elements $|G|$



example : permutation group

three standard notations for permutations

$$\rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$$
$$\pi = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132) = \text{birdtrack notation},$$

two-line notation **cycle notation** **birdtrack notation**

all mean : permute objects **1, 2, 3**, so that after the permutation

$$\pi(1) = 3, \quad \pi(2) = 1, \quad \pi(3) = 2$$



permutation multiplication

to compose permutations

$$\rho = (12) = \begin{array}{c} \text{X} \\ \text{---} \end{array}, \quad \text{followed by} \quad \pi = (132) = \begin{array}{c} \text{X} \\ \text{X} \\ \text{---} \end{array}$$

multiply cycles

$$\pi \circ \rho = (132)(12) = (1)(23) = (23)$$

omit '1' omit one-cycles

or compose diagrams

$$\pi \circ \rho = \begin{array}{c} \text{X} \\ \text{X} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{X} \\ \text{---} \end{array}$$

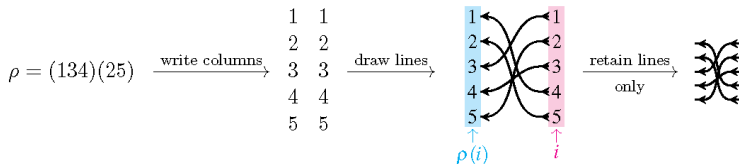
S_n : permutation group on n quarks

- ▶ from permutation cycles to birdtracks :



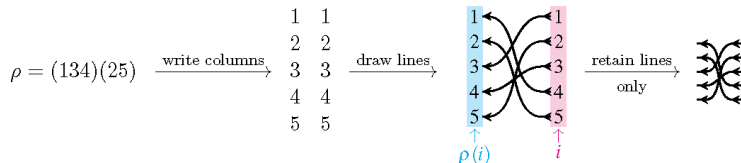
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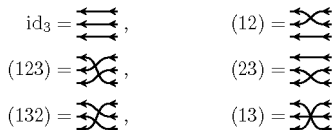


S_n : permutation group on n quarks

- ▶ from permutation cycles to birdtracks :



- ▶ permutation group S_3 elements



- ▶ birdtrack multiplication :
connect lines and straighten them out

$$\begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} \cdot \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array}$$

- ▶ birdtrack inverse

- ▶ birdtrack multiplication :
connect lines and straighten them out

$$\begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} \cdot \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \\ \leftarrow \leftarrow \end{array}$$

- ▶ birdtrack inverse ρ^{-1} of a permutation $\rho \in S_n$:

- ▶ birdtrack multiplication :
 connect lines and straighten them out

The diagram illustrates the multiplication of two birdtracks. On the left, two birdtracks are shown side-by-side, separated by a dot. The first birdtrack consists of two crossings, and the second also consists of two crossings. An equals sign follows, leading to a single birdtrack with two crossings. A second equals sign follows, leading to a single horizontal line with two crossings, representing the straightened-out result of the multiplication.

- ▶ birdtrack inverse ρ^{-1} of a permutation $\rho \in S_n$:
 reflect ρ about vertical axis, reverse the arrows

- ▶ birdtrack multiplication :
connect lines and straighten them out

$$\begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} \cdot \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} = \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array}$$

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closure, associativity, identity, inverse : it's a group !

S_3 multiplication table



definition : algebra over group elements

the vector space $\mathcal{A} = \mathbb{C}[G]$ constructed from linear combinations of group elements

$$a = \sum_g \lambda_g g, \quad g \in G, \quad \lambda_g \in \mathbb{C}$$

for example :

$$= \begin{array}{c} \text{Diagram 1} \end{array} - 5 \begin{array}{c} \text{Diagram 2} \end{array} + 2i \begin{array}{c} \text{Diagram 3} \end{array}, \quad a \in \mathbb{C}[S_4]$$

is an *algebra*

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is an *algebra*, with $\mathcal{A} \cdot \mathcal{A} \rightarrow \mathcal{A}$ distributive multiplication

$$(a + b) c = ac + bc, \quad a, b, c \in \mathcal{A}$$

compact birdtrack notation : (anti)symmetrizers

denote symmetrizers S / anti-symmetrizers A

$$S = \frac{1}{n!} \sum_{\pi \in S_n} \pi \quad \text{and} \quad A = \frac{1}{n!} \sum_{\pi \in S_n} \text{sign}(\pi) \pi$$

by white, black bars

$$S = \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \left| \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \right. \quad \text{and} \quad A = \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \left| \text{---} \right. .$$



compact birdtrack notation for algebra elements

$$S_{24} = \text{birdtrack diagram}, \quad S_{134} = \text{birdtrack diagram}, \quad A_{1234} = \text{birdtrack diagram}$$

The diagram shows three birdtrack notations for algebra elements. The first, S_{24} , is a white vertical bar with four horizontal lines passing through it; the top two lines cross each other. The second, S_{134} , is a white vertical bar with four horizontal lines; the top two lines cross each other and also cross the bar. The third, A_{1234} , is a thick black vertical bar with four horizontal lines passing through it.



birdtrack computations are compact

$$\begin{aligned}
 A &= \frac{1}{6} \left(\begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} - \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} + \begin{array}{c} \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \\ \leftarrow \leftarrow \leftarrow \end{array} \right) \\
 &= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}
 \end{aligned}$$



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$$= \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

antisymmetrize twice :

$$A^2 = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}$$

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as the antisymmetrized state is already antisymmetric

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as the antisymmetrized state is already antisymmetric

$A^2 = A$, i.e. S , A are projection operators.

Operating on what ?

what does operator A do ?

In the full index notation

$$A_{a_1 a_2 \dots a_p, b_p \dots b_2 b_1} = \frac{1}{p!} \left\{ \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \dots \delta_{a_p}^{b_p} - \delta_{a_2}^{b_1} \delta_{a_1}^{b_2} \dots \delta_{a_p}^{b_p} + \dots \right\}$$

$$A = \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \Big| \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array}$$



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$$A = \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array} \Big| \begin{array}{c} \text{---} \\ \text{---} \\ \vdots \\ \text{---} \end{array}$$

is a (tensorial) matrix that acts on p copies of vector space V

$$A : V^p \rightarrow V^p$$



practice : birdtracks \leftrightarrow indices

operation of permuting tensor indices is a linear operation,
represented by a $[d \times d]$ matrix:

$$\sigma_{\alpha}^{\beta} = \sigma_{b_1 \dots b_p}^{a_1 a_2 \dots a_q, d_p \dots d_1, c_q \dots c_2 c_1}.$$

for 2-index tensors, there are two permutations:



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for 2-index tensors, there are two permutations:

$$\begin{aligned} \text{identity:} \quad \mathbf{1}_{ab}^{cd} &= \delta_a^d \delta_b^c = \begin{array}{c} \leftarrow \\ \leftarrow \end{array} \\ \text{flip:} \quad \sigma_{(12)ab}^{cd} &= \delta_a^c \delta_b^d = \begin{array}{c} \leftarrow \\ \rightarrow \end{array}. \end{aligned}$$



practice : birdtracks \leftrightarrow indices

For 3-index tensors, there are six permutations:

$$\begin{aligned}
 \mathbf{1}_{a_1 a_2 a_3}^{b_3 b_2 b_1} &= \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array} \\
 \sigma(12)_{a_1 a_2 a_3}^{b_3 b_2 b_1} &= \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array}
 \end{aligned}$$



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 \sigma(12)_{a_1 a_2 a_3}^{b_3 b_2 b_1} &= \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} = \begin{array}{c} \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \\ \leftarrow \end{array}
 \end{aligned}$$

WOA!



- ▶ (conventional) Young operators are not Hermitian
 \rightsquigarrow do not yield orthogonal bases

$$Y \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \frac{4}{3} \begin{array}{c} \text{[white]} \text{ [black]} \\ \text{---} \\ \text{---} \end{array}, \quad Y \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \frac{4}{3} \begin{array}{c} \text{[white]} \text{ [black]} \\ \text{---} \\ \text{---} \end{array}$$

- ▶ Hermitian Young operators 😊

$$P \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} = \frac{4}{3} \begin{array}{c} \text{[white]} \text{ [black]} \text{ [white]} \\ \text{---} \\ \text{---} \end{array}, \quad P \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} = \frac{4}{3} \begin{array}{c} \text{[black]} \text{ [white]} \text{ [black]} \\ \text{---} \\ \text{---} \end{array}$$

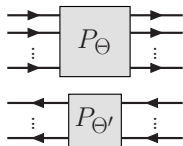
SK & M. Sjö Dahl, J. Math. Phys. **55** (2014) 021702, arXiv:1307.6147

J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051702, arXiv:1610.10088

J. Alcock-Zeilinger & H. Weigert, J. Math. Phys. **58** (2017) 051703, arXiv:1610.08802



- ▶ apply two Hermitian Young operators



- ▶ further decompose by subtracting contractions, e.g.

$$\begin{aligned}
 & \text{Diagram 1} = \frac{2}{N+1} \text{Diagram 2} \\
 & + \left(\text{Diagram 3} - \frac{2}{N+1} \text{Diagram 4} \right)
 \end{aligned}$$

The equation shows a decomposition of a diagram. On the left is a diagram with three horizontal lines, each having a small white box in the middle. This is equal to $\frac{2}{N+1}$ times a diagram with three horizontal lines, each having a white box in the middle and a loop extending from the top and bottom lines. This is then added to a term in large parentheses: a diagram with three horizontal lines and a white box in the middle, minus $\frac{2}{N+1}$ times a diagram with three horizontal lines, each having a white box in the middle and a loop extending from the top and bottom lines.



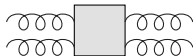
	number of multiplets		dimension of color space	
	$N = 3$	$N = \infty$	$N = 3$	$N = \infty$
$A^2 \rightarrow A^2$	6	7	8	9
$A^3 \rightarrow A^3$	29	51	145	265
$A^4 \rightarrow A^4$	166	513	3 598	14 833
$A^5 \rightarrow A^5$	1 002	6 345	107 160	1 334 961





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$$\begin{array}{cccccccc}
 \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} & \otimes & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} & = & \bullet & \oplus & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} & \oplus & \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array} \\
 8 & & 8 & & 1 & & 8 & & 8 & & 10 & & \overline{10} & & 27
 \end{array}$$

$N = 3$





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 8 & 8 & 1 & 8 & 8 & 10 & \overline{10} & 27 \\
 \end{array} \quad N = 3$$

$$\begin{array}{c}
 \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \otimes \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} = \bullet \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \end{array} \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \square \\ \hline \end{array} \\
 15 & 15 & 1 & 15 & 15 & 45 & \overline{45} & 84 & 20 \\
 \end{array} \quad N = 4$$



constructing multiplet bases

- ▶ quarks only \rightsquigarrow **Hermitian** Young operators^{*,†,‡}
- ▶ gluons only[§]
- ▶ quarks & anti-quarks
work in progress (Keppeler & Alcock-Zeilinger)
- ▶ quarks, anti-quarks & gluons \rightsquigarrow (at least) **two** strategies[¶]
work in progress (Alcock-Zeilinger & Keppeler & Sjö Dahl & Thorén)

working with multiplet bases

- ▶ download bases for 6 external partons[¶]
- ▶ software
M. Sjö Dahl: ColorMath/ColorFull (Mathematica/C++ packages)
- ▶ multiplet bases \rightsquigarrow Wigner 3j & 6j coefficients[¶]

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