Color structure of $\mathrm{SU}(N)$ QCD

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QCD birdtracks master class 2019
Saint-Jacut-de-la-Mer, France

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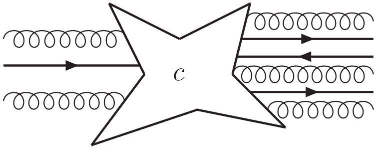
Plan

- construct bases for QCD color space
- using projectors onto $\mathrm{SU}(N)$ irreps
- written as birdtracks

- QCD process



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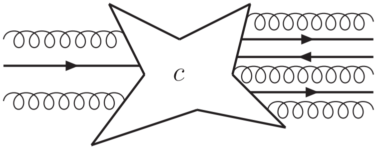

- color structure $c : \overline{V} \otimes V^2 \otimes A^3 \to V \otimes A^2$

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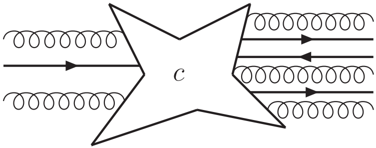

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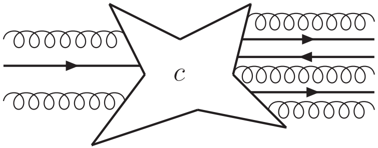

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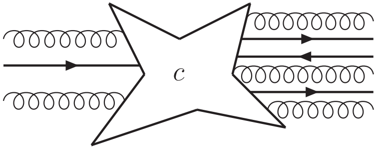

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- as linear combination of color structures such amplitudes require basis of $\underbrace{\{\text{singlets} \subset (\overline{V} \otimes V)^{n_q} \otimes A^{n_g}\}}_{\text{color space}}$

QCD $\mathrm{SU}(N)$ color basis

form all linearly independent singlets in $(\overline{V} \otimes V)^{n_q} \otimes A^{n_g}$

There are two ways of doing it :

- trace basis, any n : convert gluons $\to$ quark-antiquarks,
  construct basis for $(\overline{V} \otimes V)^{n_q+n_g}$
  restrict to n_g external gluons

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  - block-diagonalize to irreps of $\mathrm{SU}(n)$

linear transformations : index notation

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$$[n \times n] \text{ matrices } G_a{}^b \in G$$

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- conjugate multiplet : antiquark wave function as

$$q'^a = G^a{}_b \, q^b$$

linear transformations : index notation

- tensors : multi-particle states transform as $V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$

$$p'_a q'_b r'^c = G_a{}^f \, G_b{}^e \, G^c{}_d \, p_f q_e r^d$$

Note: repeated indices are always summed over

$$G_a{}^b x_b \equiv \sum_{b=1}^{n} G_a{}^b x_b \,,$$

unless explicitly stated otherwise.

three tensor notations

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- collective indices notation :

$$q'_\alpha = G_\alpha{}^\beta q_\beta \,, \qquad \alpha = \left\{ \begin{array}{c} c \\ ab \end{array} \right\} ,$$

- birdtrack notation

$$T \in V \otimes A \otimes \overline{V} \otimes A$$

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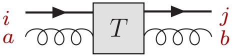
assign indices



$$= T^{i}{}_{a,\, jb}$$

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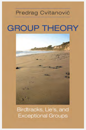

Predrag Cvitanović
Group Theory – Birdtracks, Lie's and Exceptional Groups[1]
Princeton University Press 2008

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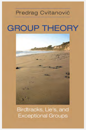

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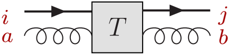

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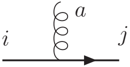
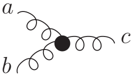
- examples



$$= (t_a)^i{}_j \ ,$$



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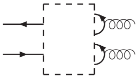
trace basis

example : color structure of 2 gluons → quark-antiquark

- attach a quark-gluon vertex to each external gluon line



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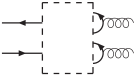
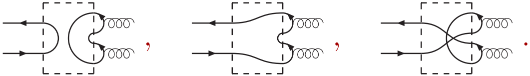


- connect all external quark lines to either external quark lines or to quark-gluon vertices



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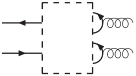

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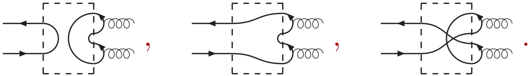
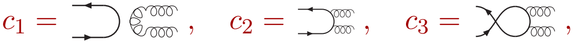
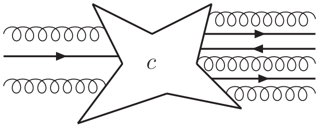


- result is a trace basis



- frequently used : trace basis



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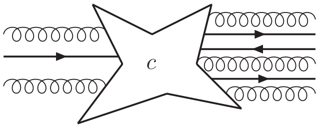
- consists of (traces of) products of generators
  - ☺ easy to construct



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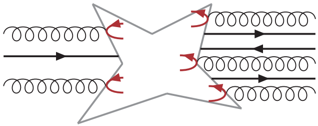

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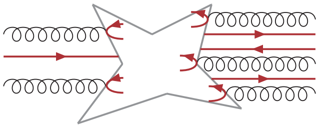

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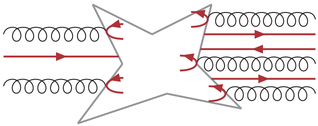

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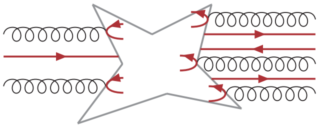

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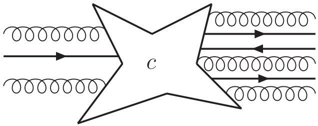
- constructed from projectors



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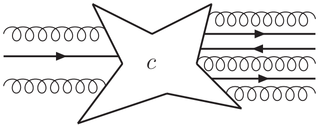
  - ☺ orthogonal & minimal



constructing multiplet bases

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- quarks only $\rightsquigarrow$ Hermitian Young operators[2,3,4]
- gluons only[5]
- quarks & anti-quarks
  work in progress (Keppeler & Alcock-Zeilinger)
- quarks, anti-quarks & gluons $\rightsquigarrow$ (at least) two strategies[5]
  work in progress (Alcock-Zeilinger & Keppeler & Sjödahl & Thorén)

working with multiplet bases

- download bases for 6 external partons[5]
- software
  M. Sjödahl: ColorMath/ColorFull (Mathematica/C++ packages)
- multiplet bases $\rightsquigarrow$ Wigner 3j & 6j coefficients[6]

[2] S. Keppeler and M. Sjödahl, J. Math. Phys. **55**, 021702 (2014).
[3] J. Alcock-Zeilinger and H. Weigert, J. Math. Phys. **58**, 051702 (2017).
[4] J. Alcock-Zeilinger and H. Weigert, J. Math. Phys. **58**, 051703 (2016).
[5] S. Keppeler and M. Sjödahl, J. High Energy Phys. **2012**, 1–49 (2012).
[6] M. Sjödahl and J. Thorén, J. High Energy Phys. **09**, 55 (2015).