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Birdtracks for SU(N)

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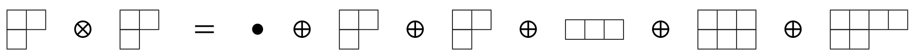
4.4 Gluon projectors

In Sec. 4.3 we have seen that the crucial ingredient for any multiplet basis are the projection operators to multiplets within $A^{\otimes n}$.

The construction rules for projectors depend on when a multiplet M appears for the first time in the sequence

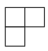
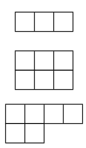
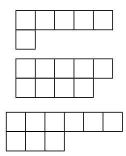
$$A^{\otimes 0} = \bullet\,, \; A^{\otimes 1} = A\,, \; A^{\otimes 2} = A \otimes A\,, \; A^{\otimes 3}\,, \; A^{\otimes 4}\,, \ldots \tag{103}$$

We call $n_f(M) = 0, 1, 2, 3, 4, \ldots$ the first occurrence of multiplet M. Consequently, the only multiplets with first occurrence 0 and 1 are the trivial and the adjoint representation, respectively, in short $n_f(\bullet) = 0$ and $n_f(A) = 1$. For SU(3), we have, e.g.,



$$8 \otimes 8 = 1 \oplus 8 \oplus 8 \oplus 10 \oplus \overline{10} \oplus 27\,, \tag{104}$$

and, consequently, the decuplets and the 27-plet have first occurrence 2. The following table shows some more SU(3) examples.

n_f	0	1	2	3
SU(3) Young diagrams	$\bullet =$ 	$A =$ 		

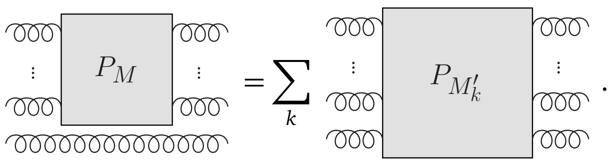
[4]Using the terminology of Sec. 4.4 only projectors to old multiplets have to be constructed.

The first occurrence of any multiplet can in principle be determined by repeatedly multiplying Young diagrams for the adjoint representation until the desired multiplet appears. One can also derive [6, App. B] a graphical rule for directly determining $n_f(M)$ from the corresponding Young diagram.

Our construction of gluon projectors will be recursive. Assume that we have determined the projectors for the decomposition of $A^{\otimes(n-1)} = \bigoplus_j M_j$ into multiplets M_j. In order to decompose $A^{\otimes n}$ we have to multiply each $M \subseteq A^{\otimes(n-1)}$ with another A, i.e. we consider

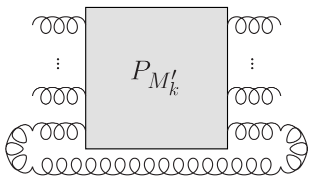
$$M \otimes A = \bigoplus_k M'_k\,, \tag{105}$$

which for the projectors reads

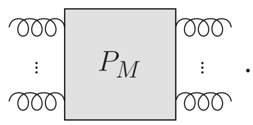


(106)

As for the Hermitian Young opeators, cf. the discussion around Eqs. (90)–(92), our projectors will be such that $\forall\, k$



is proportional to



(107)

For the decomposition (105) one can show [6] that

(i) $n_f(M'_k) = n_f(M)-1$, $n_f(M)$ or $n_f(M)+1$, and

(ii) only M itself can appear with multiplicity greater than one within $M \otimes A$ (in fact it can appear up to $N-1$ times), all other multiplets are unique.

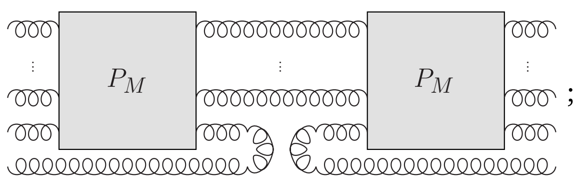
Reexamining Eq. (104), where on the l.h.s. we identify $\ydiagram{2,1} \otimes \ydiagram{2,1} = M \otimes A$, we can verify both statements: Property (i) is trivially true since $n_f(M) = 2$, but we also see that (ii) holds, as on the r.h.s. all multiplets except for $M = \ydiagram{2,1}$ appear only once, and $M = \ydiagram{2,1}$ itself appears twice (which here is the maximum degeneracy since $N = 3$).

We call a multiplet $M \subseteq A^{\otimes n}$ old if $n_f(M) < n$ and new if $n_f(M) = n$. Construction rules for projectors onto multiplets $M'_k \subseteq M \otimes A$ depend on whether M and M' are old or new and on which of the cases (i) we have at hand. We now give some examples for important cases, which all appear in $A \otimes A$; the complete set of construction rules (and their proofs) are given in [6].

M new, $n_f(M'_k) = n_f(M)-1$

Write down P_M twice and bend back the last gluon line,

$$P_{M'_k} = \frac{\dim M'_k}{\dim M}$$



(108)

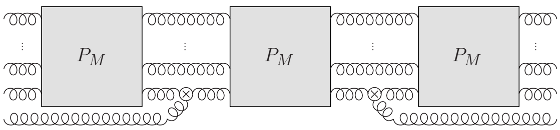
the prefactor makes sure that $P_{M'_k}^2 = P_{M'_k}$. Example: The projector to the singlet in eq. (104) is constructed in this way,

$$P_{\bullet} = \frac{1}{N^2-1} \qquad . \tag{109}$$

$n_f(M'_k) = n_f(M)$, M'_k equivalent to M

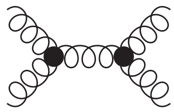
Write down P_M twice and and connect the new gluon line to one of the old gluon lines by means of two f- or d-vertices (or a linear combination of them),

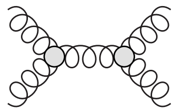
$$P_{M'_k} = \gamma \qquad , \tag{110}$$



where $\otimes$ is to be replaced by $\bullet$ or $\circ$ (or a linear combination). A formula for the normalisation factor γ is given in [6]. Examples: The two projectors to copies of the adjoint representation on the r.h.s. of Eq. (104) are constructed in this way,

$$P_{Aa} = \frac{1}{2NT_R} \qquad \text{and} \qquad P_{As} = \frac{N}{2(N^2-4)T_R} \qquad . \tag{111}$$

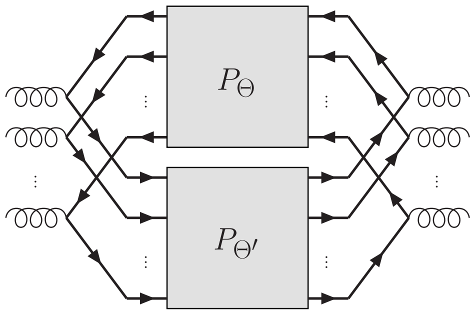




New multiplets

Here we use that $A \subset \overline{V} \otimes V$: Split each gluon line into a quark and an anti-quark line by means of a generator (quark-gluon-vertex). Then put a Young operator (Hermitian or not doesn't matter) on the quark lines and on the anti-quark-lines (one each),

$$T = \qquad . \tag{112}$$



One can show [6] that for $\Theta, \Theta' \in \mathscr{Y}_n$ the tensor product $\overline{\Theta} \otimes \Theta'$ contains exactly one new multiplet of the decomposition of $A^{\otimes n}$. Since it also contains contributions from other (old) multiplets, these have to be removed in a Gram-Schmidt step,

$$\widetilde{T} = T - \sum_{M \text{ old}} \frac{\operatorname{tr}(P_M T)}{\dim M} P_M \,. \tag{113}$$

Finally, we obtain the desired projector onto the new multiplet by normalising $\widetilde{T}$,

$$P_{M'_k} = \frac{\dim M'_k}{\operatorname{tr} \widetilde{T}} \widetilde{T} \,. \tag{114}$$

Examples

Choosing $(\yng(2),\yng(2))$ for the pair (Θ,Θ') this procedure leads to the projector onto the 27-plet in Eq. (104). Projectors onto the two decuplets in Eq. (104) we find by choosing $(\yng(2),\yng(1,1))$ and $(\yng(1,1),\yng(2))$. The resulting formulae are, e.g., given in [1, Table 9.4], [7, App. A.1] or [6, Eq. (1.23)].

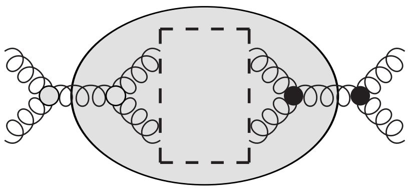
Note that all birdtrack construction rules in this section never use that $N = 3$. Thus, the projection operators constructed above, project onto multiplets within $A\otimes A$ for any N. However, for $N \geq 4$ there is exactly one more multiplet in the decomposition of $A\otimes A$. We find the corresponding projector by applying the construction rules for new multiplets, this time using $(\Theta,\Theta') = (\yng(1,1),\yng(1,1))$; the result can be shown to vanish for $N = 3$. This is a general feature of these birdtrack constructions for gluon projectors (and for multiplet bases): The construction rules are independent of N. If for small N there are fewer multiplets (or colour spaces of smaller dimension) then some of the terms simply vanish – as opposed to less obvious linear dependencies, which appear in trace bases, cf. Eq. (86).

Exercise 20 *Construct the projectors $P_{\yng(3)}$, $P_{\yng(3,3)}$ and $P_{\yng(4,2)}$ according to the rules given above. You may either use $\overline{P_{\yng(3)}} = P_{\yng(3,3)}$ in your calculations or verify this property from your result. Also determine the dimensions of the corresponding multiplets for arbitrary N.*

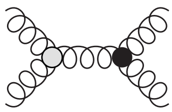
4.5 Some multiplet bases

$A^{\otimes 4}$

The multiplet basis for the colour space within $A^{\otimes 4}$ is given by the projection operators – six for $N = 3$, seven for $N \geq 4$ – normalised according to Eq. (102), and two transition operators mapping the two multiplets carrying the adjoint representation onto each other. The latter we construct by writing down projectors for each of the two copies (ignoring prefactors),

 , (115)

and then seeking a non-vanishing connection inside the dashed rectangular box. We do not have to explicitly write out such a connection; simply notice that when we have found one then the expression inside the grey ellipse is an invariant tensor mapping A to A and thus, according to Schur's lemma, it is proportional to . Hence, the first transition operator is proportional to

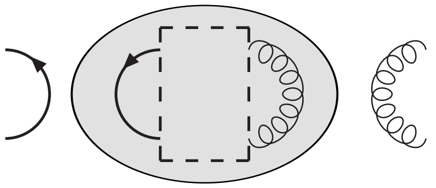
 . (116)

We obtain the second transition operator by interchanging $\bullet$ and $\circ$.

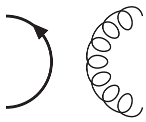
Exercise 21 *Normalise the transition operator* (116).

$\overline{V} \otimes V \otimes A^{\otimes 2}$

The quark-anti-quark pair can either be in a singlet state or in the adjoint representation. If it is in a singlet state we need to find a transition operator mapping the singlet within $A \otimes A$ to the singlet within $\overline{V} \otimes V$. To this end we write down the two projectors (ignoring prefactors) and seek a non-vanishing connection inside the dashed box,

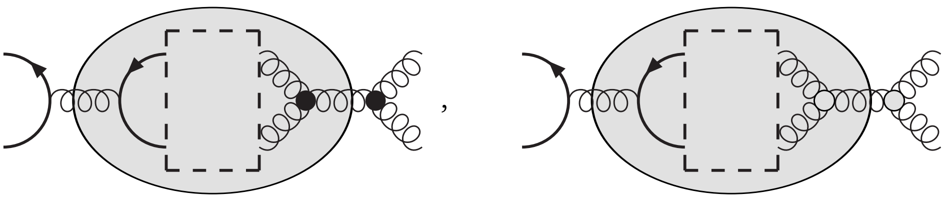
 . (117)

No matter what this connection looks like, the part inside the grey ellipse is just a number, i.e. the desired transition operator is proportional to

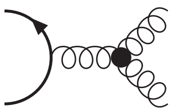
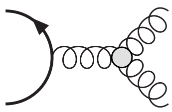
 . (118)

Exercise 22 *Find a non-vanishing way to connect the lines withing the dashed box in diagram* (117) *and evaluate the resulting term inside the grey ellipse.*

If the quark-anti-quark pair is in the adjoint representation, then we need to find transition operators to the two adjoint representations within $A \otimes A$. Once more we write down the corresponding projectors (omitting prefactors), and seek non-vanishing connections inside the dashed boxes,

 . (119)

Once more, we do not have to find these connections explicitly, but simply notice that the parts within the grey ellipses have to be proportional to , i.e. the transition operators are proportional to

 and  . (120)

The three tensors in (118) and (120) form an orthogonal multiplet basis for the colour space within $\overline{V} \otimes V \otimes A^{\otimes 2}$, which is to be compared to the non-orthogonal trace basis (83).

Exercise 23 *Normalise the basis vectors in* (118) *and* (120).