Birdtracks for SU(N)

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4.2 Multiplet bases

For the construction of a multiplet basis for $c \in (V \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$ we consider c as a linear map, say

$$c : (\overline{V} \otimes V)^{\otimes k_q} \otimes A^{\otimes k_g} \to (\overline{V} \otimes V)^{\otimes (n_q - k_q)} \otimes A^{\otimes (n_g - k_g)} , \tag{88}$$

for some $0 \leq k_q \leq n_q$ and $0 \leq k_g \leq n_g$. In general, we thus have a linear map $c : W_1 \to W_2$, between two vector spaces, carrying representations Γ_1 and Γ_2 of SU(N). Moreover, c being an invariant tensor means that

$$c \circ \Gamma_1(g) = \Gamma_2(g) \circ c \quad \forall\, g \in \mathrm{SU}(N) . \tag{89}$$

Now we are in a situation where we can employ Schur's lemma. It is often formulated for the case where W_1 and W_2 carry irreducible representations saying that

- if the two representations are inequivalent, then c vanishes identically, and
- if the two representations are equivalent and $W_1 = W_2$, then c is a multiple of the identity.

In our case the representations are typically not irreducible, and then Schur's lemma implies that c can only map subspaces onto each other that carry the same irreducible representation, i.e. c maps only equivalent multiplets onto each other.

Now consider the case when $W_1 = W_2 =: W$. If we decompose W into multiplets, i.e. into irreducible SU(N)-invariant subspaces, then the projectors onto these multiplets are distinguished elements of colour space.

	Number of multiplets		Dimension of colour space	
	$N=3$	$N=\infty$	$N=3$	$N=\infty$
$A^{\otimes 2} \to A^{\otimes 2}$	6	7	8	9
$A^{\otimes 3} \to A^{\otimes 3}$	29	51	145	265
$A^{\otimes 4} \to A^{\otimes 4}$	166	513	3 598	14 833
$A^{\otimes 5} \to A^{\otimes 5}$	1 002	6 345	107 160	1 334 961

Table 1: Number of projection operators and dimension of the colour space within $A^{\otimes(2n)}$, for colour structures viewed as maps $A^{\otimes n} \to A^{\otimes n}$. The first two columns show the number of multiplets (counted with multiplicities) in the decomposition of $A^{\otimes n}$, both for $N = 3$ and for $N \geq n$. The last two columns contain the dimensions of the respective colour spaces; the dimension in the last column is also equal to the number of elements of the corresponding trace basis.

- If each multiplet appears only once in the decomposition of W then the projectors form a basis of colour base. If, moreover, the projectors are Hermitian, then this basis is orthogonal.
- If some multiplets in the decomposition of W have a multiplicity > 1 then we have to complement the projectors with operators mapping equivalent multiplets onto each other.

In practice, finding the multiplets in the decomposition of W and their multiplicities can, e.g., be done by multiplying Young diagrams according to the standard rules. The crucial step is then to find Hermitian projectors onto these multiplets. Finally, multiplet bases can be constructed straightforwardly from Hermitian projection operators.

In the following sections we discuss how to construct Hermitian projection operators as well as multiplet bases for the cases $V^{\otimes n} \to V^{\otimes n}$ and $A^{\otimes n} \to A^{\otimes n}$. Moreover, we will see that multiplet cases for any colour space can be constructed from projectors for $A^{\otimes n} \to A^{\otimes n}$.

Comparison

Trace bases are convenient since they are easy to construct and since there is a simple algorithm for expanding arbitrary colour factors into a trace basis. In general, trace bases are overcomplete, i.e. expansions tend to have too many terms. For instance, the trace basis for the colour space within $A^{\otimes n}$ is a proper basis if $n \leq N$ but for $n > N$ it is only a spanning set – the basis vectors are linearly dependent. Trace bases, typically, are also non-orthogonal.

Constructing multiplet bases requires more work than constructing trace bases. In return we obtain not only a proper basis, i.e. the basis vectors are linearly independent, but also an orthogonal basis. Even though the dimension of colour space depends on N, the number of colours, the birdtrack construction of multiplet bases can be carried out independently of N, and then for small N some basis vectors simply vanish.

The numbers in Table 1 give us an impression of the potential advantage of multiplet bases over trace bases. Imagine doing a calculation for $N = 3$ with 6 to 10 external gluons involved. Then the number of trace basis elements exceeds the dimension of colour space by, roughly, a factor of 2 to 12. Expanding colour structures in a trace or multiplet basis and then, e.g., calculating scalar products will result in 4 to 144 times as many terms when using a trace basis instead of a multiplet basis.

4.2 Multiplet bases for quarks

We first consider the case without external gluons, i.e. we are interested in the colour space within $(\overline{V} \otimes V)^n$. Tensors $c \in (\overline{V} \otimes V)^{\otimes n}$ can be viewed as linear maps $c : V^{\otimes n} \to V^{\otimes n}$, and

Young operators Y_Θ project onto multiplets. Unfortunately, Young operators are in general not Hermitian, as can be seen by, e.g. inspecting Eq. (74): Mirroring and reversing the arrows does not yield back the original expression.

However, Hermitian operators P_Θ corresponding to standard Young tableaux Θ can be constructed. In [1] they are derived as solutions of certain characteristic equations. They can also be written down directly starting from a Young tableaux as follows. Consider the sequence of Young tableaux $\Theta_j \in \mathscr{Y}_j$ obtained from $\Theta \in \mathscr{Y}_n$ by, step by step, removing the box with the highest number, e.g., starting with $\Theta = \Theta_3 = \ytableaushort{12,3} \in \mathscr{Y}_3$ we obtain

$$\Theta_1 = \ytableaushort{1}\,, \quad \Theta_2 = \ytableaushort{12}\,, \quad \Theta_3 = \ytableaushort{12,3}\,. \tag{90}$$

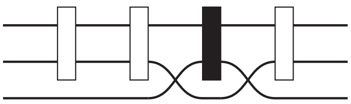
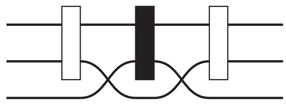
Young operators for $n = 2$ are Hermitian – they are just total (anti-)symmetrisers – so we set

$$P_{\Theta_j} = Y_{\Theta_j} \quad \forall\, j \leq 2\,. \tag{91}$$

Then we define recursively

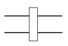
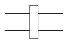
$$P_{\Theta_j} = (P_{\Theta_{j-1}} \otimes \mathbb{1}_V) Y_{\Theta_j} (P_{\Theta_{j-1}} \otimes \mathbb{1}_V) \quad \forall\, j \geq 3\,, \tag{92}$$

i.e. in birdtrack notation we take the Young operator Y_{Θ_j} and write the Hermitian Young operator $P_{\Theta_{j-1}}$ over the first $j-1$ lines, to the left and to the right. For instance,

$$P_{\ytableaushort{12,3}} = \frac{4}{3}$$  $$= \frac{4}{3}$$  , (93)

which is manifestly Hermitian, and in birdtracks it is also easy to see that

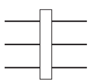
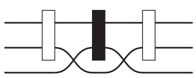
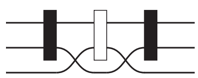
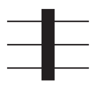
$$\operatorname{tr} P_{\ytableaushort{12,3}} = \operatorname{tr} Y_{\ytableaushort{12,3}} \tag{94}$$

since ()^2 =  .

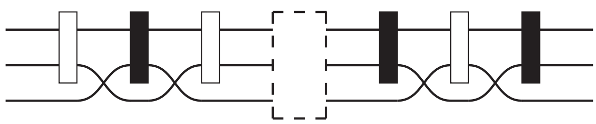
It can be shown [3] that the resulting P_Θ not only project onto the correct multiplets but that they are also Hermitian and thus mutually orthogonal with respect to the scalar product (80). Furthermore, using the Hermitian Young operators P_Θ automatically cures the loss of transversality mentioned at the end of Sec. 3.2

The recursive construction can produce initially lengthy expressions which can often be simplified considerably, see, e.g., the step-by-step example for $\ytableaushort{135,24}$ in the Appendix of [3]. Similar simplifications can be shown to occur much more generally [4] and they can be used to devise a recipe for directly writing down fully simplified Hermitian Young operators [5].

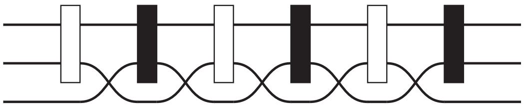
With the Hermitian Young operators

$P_{\ytableaushort{123}} =$  , $P_{\ytableaushort{12,3}} = \frac{4}{3}$  , $P_{\ytableaushort{13,2}} = \frac{4}{3}$  , $P_{\ytableaushort{1,2,3}} =$  , (95)

we have completely decomposed $V^{\otimes 3}$ into an orthogonal sum of multiplets. However, the P_Θ alone do not form a basis for the colour space within $(\overline{V} \otimes V)^{\otimes 3}$, since the multiplet  appears twice, i.e. we also need an operator mapping these two multiplets onto each other. To this end we write down $P_{\ytableaushort{12,3}}$ and $P_{\ytableaushort{13,2}}$ next to each other (omitting prefactors),

 , (96)

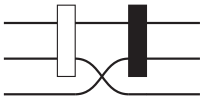
and seek a way of connecting the lines within the dashed box such that the whole expression does not vanish, because then it is guaranteed, that the resulting expression has the same kernel as $P_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}}$ and the same image as $P_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}}$. The only such connection (up to a sign) is



$$, \qquad (97)$$

and by expanding the central (anti-)symmetrisers one can verify that this expression is proportional to

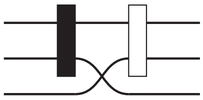
$$T_1 :=$$



$$. \qquad (98)$$

Thus, we have found a basis vector mapping $\begin{smallmatrix}1&3\\2&\end{smallmatrix}$ to $\begin{smallmatrix}1&2\\3&\end{smallmatrix}$. The vector for the reverse mapping can be obtained in the same way and reads

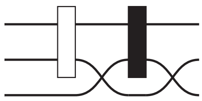
$$T_2 :=$$



$$. \qquad (99)$$

Exercise 18 *Define*

$$B =$$



$$, \qquad (100)$$

and show that B^2 is proportional to B by expanding the central (anti-)symmetrisers. Explain why this implies that the birdtrack diagram (97) is proportional to T_1.

The multiplet basis for $V^{\otimes 3} \to V^{\otimes 3}$ consisting of four Hermitian Young operators and two transition operators is orthogonal. If desired the basis vectors can be normalised: For Hermitian projection operators we generally have

$$\langle P_\Theta, P_\Theta \rangle = \mathrm{tr}(P_\Theta^\dagger P_\Theta) = \mathrm{tr}(P_\Theta P_\Theta) = \mathrm{tr}(P_\Theta) = \dim M_\Theta \qquad (101)$$

where M_Θ is the multiplet to which P_Θ projects. Hence,

$$\frac{1}{\sqrt{\dim M_\Theta}} P_\Theta \qquad (102)$$

has norm one. The transition operators can straightforwardly be normalised by a direct calculation.

Exercise 19 *Calculate $\langle T_1, T_1 \rangle$ and normalise T_1 accordingly.*