

Birdtracks for $SU(N)$

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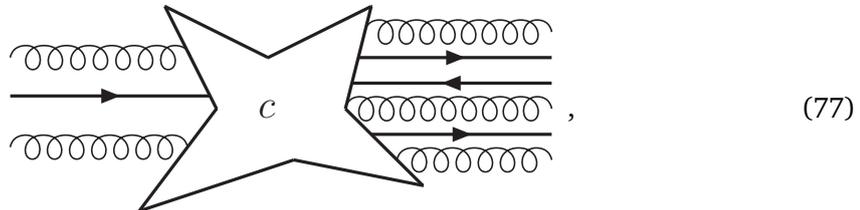
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doi:10.21468/SciPostPhysLectNotes.3

4 Colour space

Consider a QCD process with a couple of incoming and outgoing quarks, anti-quarks, and gluons. A corresponding Feynman diagram might look like this,



where inside some of the lines are connected directly, whereas others are tied together according to the QCD Feynman rules, i.e. by quark-gluon vertices as well as by 3-gluon and 4-gluon vertices. Note that these rules ensure that there are always as many (anti-)quark lines with arrows pointing away from the central blob as there are lines with arrows pointing towards it. The amplitude corresponding to such a diagram is the product of a kinematic factor (some often initially divergent momentum space integral), a spin factor and a colour factor. Here, we are only interested in the colour factor, the so-called colour structure.

The colour structure is a tensor $c \in (\bar{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$. Since colour is confined the only relevant colour structures are singlets, i.e. tensors c which transform in the trivial representation of $SU(N)$. In other words, c is a so-called invariant tensor. The singlet-subspace of $(\bar{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$ is called colour space.

Quark lines, gluon lines and quark-gluon vertices (generators) are examples of invariant tensors. Any combination (tensor product, contraction) of invariant tensors is also an invariant tensor. Thus, Young operators and triple gluon vertices (structure constants),

$$= \frac{2}{T_R} \text{ [diagram with quark line, gluon loop, and four-gluon vertex] }, \tag{78}$$

cf. Eq. (38), are invariant tensors, and also the fully symmetric

$$:= \frac{2}{T_R} \text{ [diagram with quark line, gluon loop, and four-gluon vertex] }. \tag{79}$$

The vector space $V \cong \mathbb{C}^N$, which carries the defining representation of $SU(N)$, is endowed with a scalar product, which is also left invariant by $SU(N)$. This scalar product also induces scalar products on \bar{V} , A and on arbitrary tensor products of V , \bar{V} and A . For two colour structures $c_1, c_2 \in (\bar{V} \otimes V)^{\otimes n_q} \otimes A^{\otimes n_g}$ we have

$$\langle c_1, c_2 \rangle = \text{tr}(c_1^\dagger c_2) = \left(((c_1)^{ab\dots})^{jk\dots}_{\ell m\dots} \right)^* ((c_2)^{ab\dots})^{jk\dots}_{\ell m\dots}, \tag{80}$$

where in birdtrack notation Hermitian conjugation corresponds to mirroring the diagram across a vertical line and reversing all arrows.

For calculations it is convenient to expand colour structures into a basis of colour space. The most popular bases in use are so-called trace bases (which in general are overcomplete, i.e. they are no proper bases but only spanning sets). In general, trace bases are not orthogonal. In the remainder of this section, after a brief review of trace bases, we will discuss the construction of minimal, orthogonal bases, so-called multiplet bases.

4.1 Trace bases vs. multiplet bases

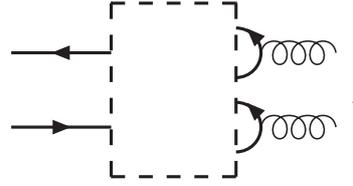
Trace bases

For diagrams with a given number of external (anti-)quark lines and gluon lines, the trace basis can be constructed as follows:

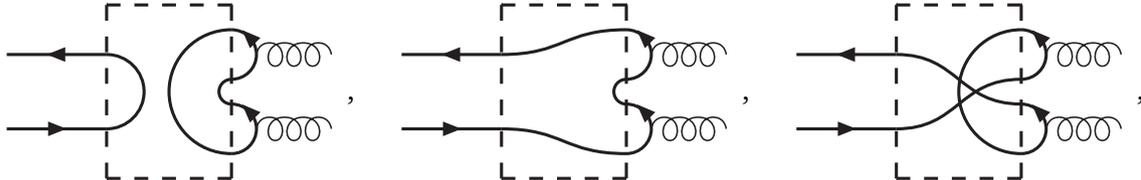
- Attach a quark-gluon vertex to each external gluon line.
- Then connect all external (anti-)quark lines to either external (anti-)quark lines or (anti-)quark lines from quark-gluon vertices.

Examples:

For $\bar{V} \otimes V \otimes A^{\otimes 2}$ we have to connect the (anti-)quark lines ending on the dashed box,


(81)

in all possible ways. The non-zero possibilities are


(82)

yielding

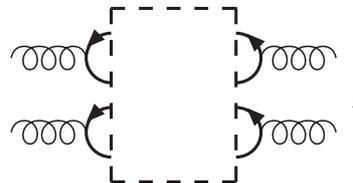
$$c_1 = \text{diagram 1}, \quad c_2 = \text{diagram 2}, \quad c_3 = \text{diagram 3}, \quad (83)$$

where we have omitted irrelevant prefactors. Note that in general the c_j are not mutually orthogonal, e.g.

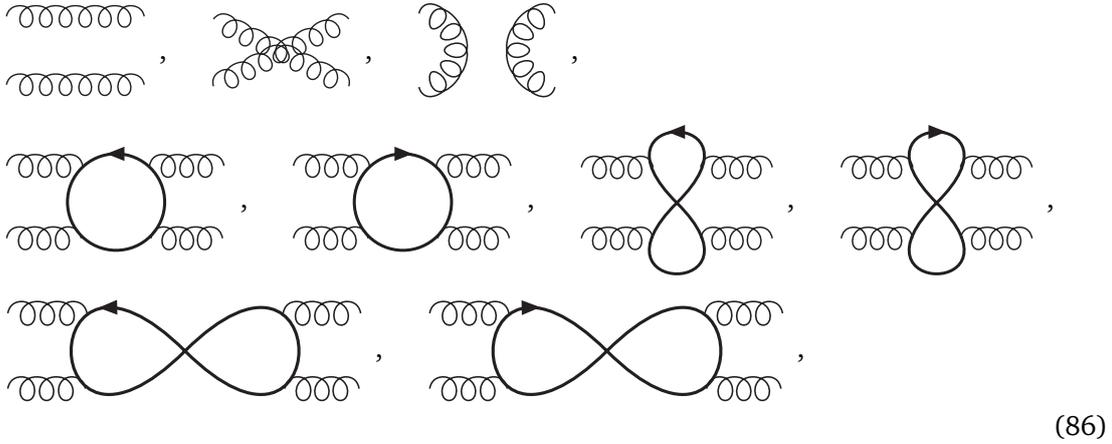
$$\langle c_1, c_2 \rangle = \text{tr}(c_1^\dagger c_2) = \text{diagram} = T_R(N^2 - 1) = C_F N, \quad (84)$$

see Eqs. (42) and (43).

For $A^{\otimes 4}$, connecting the (anti-)quark lines ending on the dashed box,


(85)

in all possible ways, we in turn find the non-vanishing diagrams



(86)

again omitting all prefactors. Written in index (or matrix) notation these basis vectors consist of traces of products of generators, thus the name trace basis. The colour structures (86) are again not mutually orthogonal, e.g.

$$\left\langle \begin{array}{c} \text{---} \\ \text{---} \end{array}, \begin{array}{c} \text{---} \\ \text{---} \end{array} \right\rangle = \text{---} = N^2 - 1. \quad (87)$$

Moreover, for $N = 3$ the nine $A^{\otimes 4}$ -vectors (86) are not linearly independent, since, as we will see below, the colour space which they span is only eight-dimensional.

There is a simple algorithm, for writing arbitrary colour factors as linear combinations of trace basis elements: First replace all four-gluon vertices by (one gluon-contracted linear combinations of) three-gluon vertices. Then replace all three-gluon vertices by (linear combinations of) quark loops with three gluons attached, see Eqs. (38) or (78). Finally, remove all internal gluon lines using Eq. (42).