Birdtracks for $\mathrm{SU}(N)$

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3.2 Young operators

Young diagrams are arrangements of n boxes in r rows of non-increasing lengths. A Young tableau Θ is a Young diagram with each of the numbers $1, \ldots, n$ written into one of its boxes. For a so-called standard Young tableau the numbers increase within each row from left to right and within each column from top to bottom. We denote the set of all standard Young tableaux with n boxes by $\mathscr{Y}_n$, e.g.

$$\mathscr{Y}_2 = \left\{ \ytableaushort{12}, \ytableaushort{1,2} \right\}, \quad \mathscr{Y}_3 = \left\{ \ytableaushort{123}, \ytableaushort{12,3}, \ytableaushort{13,2}, \ytableaushort{1,2,3} \right\}. \tag{67}$$

Removing the box containing the number n from $\Theta \in \mathscr{Y}_n$ we obtain a standard tableau $\Theta' \in \mathscr{Y}_{n-1}$.

For $\Theta \in \mathscr{Y}_n$ let $\{h_\Theta\}$ be the set of all horizontal permutations, i.e. $h_\Theta \in S_n$ leaves the sets of numbers appearing in the same row of Θ invariant. Analogously, vertical permutations v_Θ leave the sets of numbers appearing in the same column of Θ invariant. The Young operator Y_Θ is then defined in terms of the row symmetrizer, $s_\Theta = \sum_{\{h_\Theta\}} h_\Theta$, and the column anti-symmetrizer, $a_\Theta = \sum_{\{v_\Theta\}} \operatorname{sign}(v_\Theta) v_\Theta$, as

$$Y_\Theta = \tfrac{1}{|\Theta|} s_\Theta a_\Theta \,. \tag{68}$$

Note that as opposed to Eq. (57) we have not included normalising factorials. The normalisation factor is given by the product of hook lengths of the boxes of Θ, and thus depends only the shape of the Young tableau, i.e. on the Young diagram. The hook length of a given box counts the number of boxes below and to the right of this box, adding one for the box itself. For illustration we write the hook lengths into the boxes of a couple of Young diagrams,

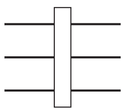
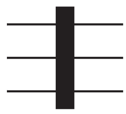
$$\young(21)\,,\quad \young(31,1)\,,\quad \young(3,2,1)\,,\quad \young(32,21)\,,\quad \young(431,21)\,, \tag{69}$$

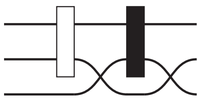
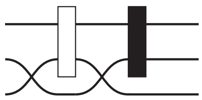
and calculate the corresponding normalisation factors,

$$|\yng(2)| = 2\,,\quad |\yng(2,1)| = 3\,,\quad |\yng(1,1,1)| = 6\,,\quad |\yng(2,2)| = 12\,,\quad |\yng(3,2)| = 24\,. \tag{70}$$

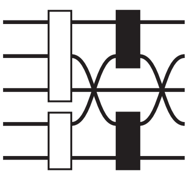
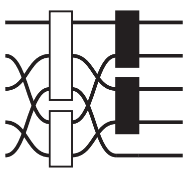
Young operators $Y_\Theta \in \mathscr{A}(S_n)$ corresponding to standard Young tableaux are primitive idempotents. For $\Theta, \vartheta \in \mathscr{Y}_n$ they satisfy $Y_\Theta Y_\vartheta = 0$, i.e. they are transversal, if the corresponding Young diagrams have different shapes. Unfortunately, for different Young tableaux of the same shape it can happen that $Y_\Theta Y_\vartheta \neq 0$ when $n > 4$.

In birdtrack notation we can draw Young operators, using partial (anti-)symmetrisers as introduced in Eq. (60), e.g.

$Y_{\young(123)} =$  , $\quad Y_{\young(1,2,3)} =$  ,

$Y_{\young(12,3)} = \frac{4}{3}$  , $\quad Y_{\young(13,2)} = \frac{4}{3}$  . (71)

Note that the normalisation factors are in agreement with Eq. (68) and the normalisation (57) of (anti-)symmetrisers. The following 5-box examples illustrate the loss of transversality for $n > 4$,

$Y_{\young(123,45)} = 2$  , and $\quad Y_{\young(135,24)} = 2$  , (72)

as we have

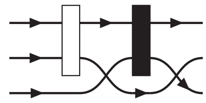
$$Y_{\young(135,24)} Y_{\young(123,45)} = 0 \quad \text{but} \quad Y_{\young(123,45)} Y_{\young(135,24)} \neq 0\,. \tag{73}$$

Exercise 16 *Verify Eq. (73).*

3.3 Young operators and SU(N): multiplets

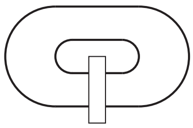
Recall that $V \cong \mathbb{C}^N$ (or $\overline{V}$) carries the defining (or complex conjugate) representation of SU(N). A tensor product $V^{\otimes n}$ (or $\overline{V}^{\otimes n}$) carries a product representation of SU(N). This tensor product also carries a representation of S_n. The representations of these two groups commute, since SU(N) acts only on individual factors, on each in the same way, whereas S_n acts by permuting the factors. In fact, it is a standard result, that Young operators viewed as linear maps $V^{\otimes n} \to V^{\otimes n}$ (or $\overline{V}^{\otimes n} \to \overline{V}^{\otimes n}$) project onto irreducible SU(N)-invariant subspaces.

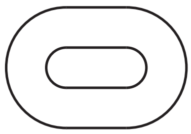
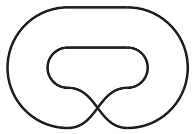
In birdtrack notation this means that we simply add arrows to the lines in all diagrams for Young operators, all pointing in the same direction, e.g., $Y_{\substack{1\,2\\3}} : V^{\otimes n} \to V^{\otimes n}$ reads

$$Y_{\substack{1\,2\\3}} = \frac{4}{3}$$  , (74)

and since all arrows point in the same direction, we usually immediately drop them again.

We refer to irreducible SU(N)-invariant subspaces as multiplets, e.g., a one-dimensional subspace (carrying the trivial representation) is called singlet. The dimension of a multiplet is given by the trace of the projector, e.g.

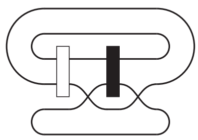
$$\operatorname{tr} Y_{\substack{1\,2}} =$$ 

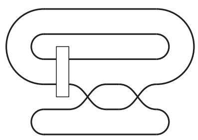
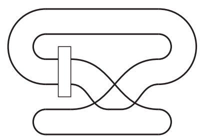
$$= \frac{1}{2}\Bigg($$  $+$  $\Bigg)$

$$= \frac{1}{2}(N^2 + N)$$

$$= \frac{N(N+1)}{2}, \tag{75}$$

or

$$\operatorname{tr} Y_{\substack{1\,2\\3}} = \frac{4}{3}$$ 

$$= \frac{2}{3}\Bigg($$  $-$  $\Bigg)$

$$= \frac{1}{3}\big(N^2(N+1) - N(N+1)\big)$$

$$= \frac{N}{3}(N^2 - 1), \tag{76}$$

For $N = 3$ the latter describes an octet, which actually carries the adjoint representation of SU(3).