Birdtracks for SU(N)

Stefan Keppeler

Fachbereich Mathematik, Universität Tübingen,
Auf der Morgenstelle 10, 72076 Tübingen, Germany

⋆ stefan.keppeler@uni-tuebingen.de

Abstract

I gently introduce the diagrammatic birdtrack notation, first for vector algebra and then for permutations. After moving on to general tensors I review some recent results on Hermitian Young operators, gluon projectors, and multiplet bases for SU(N) color space.

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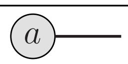
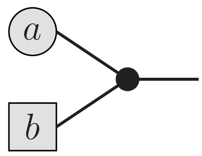
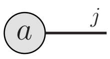
1 Birdtracks, SO(3) example

We begin with vectors $\vec{a}, \vec{b}, \vec{c}, \ldots \in \mathbb{R}^3$, scalar and cross products, review the index notation and introduce the diagrammatic birdtrack notation as illustrated in the following table.

	index notation	birdtrack notation
vector $\vec{a}$	a_j	
scalar product $\vec{a} \cdot \vec{b}$	$a_j b_j$	
cross product $\vec{a} \times \vec{b}$	$\varepsilon_{jk\ell} a_j b_k$	

More precisely, translating back and forth between index notation and birdtracks is achieved by assigning indices to external lines,

$$a_j =$$  . (1)

Index contractions (we always sum over repeated indices) correspond to joining lines,

$$a_j b_j =$$  . (2)

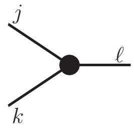
Consequently, an isolated line is a Kronecker-δ,

$$\delta_{jk} =$$  , (3)

(contracting with a Kronecker-δ corresponds to extending a line). For the totally anti-symmetric ε,

$$\begin{aligned} \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} &= 1 \\ \varepsilon_{213} = \varepsilon_{132} = \varepsilon_{321} &= -1 \\ \varepsilon_{jk\ell} &= 0 \quad \text{if at least two indices have the same value,} \end{aligned} \tag{4}$$

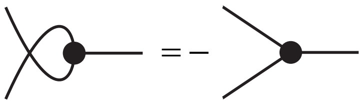
we write a vertex,

$$\varepsilon_{jk\ell} =$$ 

, (5)

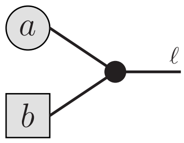
thereby agreeing to read off indices in counter-clockwise order.

When not assigning indices – which is what we want to do most of the time – the position where an external line ends determines which lines have to be identified in equations. For instance, the anti-symmetry of ε in the first two indices is expressed as



. (6)

Now we can use this notation to write components of cross products,

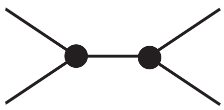
$$\varepsilon_{jk\ell} a_j b_k =$$ 

, (7)

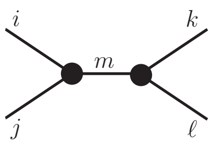
where we omit labelling lines with indices over which we sum anyway. Equivalently, omitting indices altogether,

$$\vec{a} \times \vec{b} = \quad . \tag{8}$$

Let's study the following diagram,


. (9)

Assigning indices for a moment,

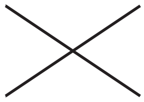


. (10)

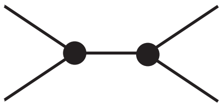
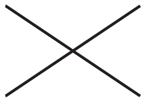
we see that the only non-zero terms have

$$\begin{gathered} i = k \text{ and } j = \ell, \text{ or} \\ i = \ell \text{ and } j = k, \end{gathered}$$

since otherwise we cannot satisfy $i \neq j \neq m \neq i$ and $m \neq \ell \neq k \neq m$. Thus, (9) is a linear combination of the diagrams

 and  . (11)

Closer inspection shows

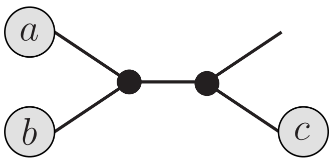
 =  −  , (12)

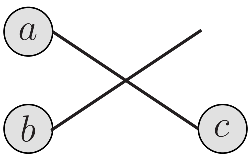
which is nothing but the birdtrack version of the well-known identity

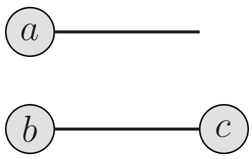
$$\varepsilon_{ijm}\varepsilon_{\ell km} = \delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}\,. \tag{13}$$

(Of course, we could have also obtained Eq. (12) by translating Eq. (13) into birdtrack notation instead of deriving the identity within birdtrack notation.)

Equation (12) can be used in order to derive identities for double cross products and similar formulas from vector algebra which are notoriously difficult to remember. For instance,

$(\vec{a} \times \vec{b}) \times \vec{c} =$ 

$=$ 

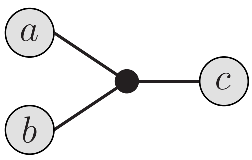
$-$ 

$= (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}\,.$ (14)

Exercise 1 *Derive a similar identity for* $(\vec{a} \times \vec{b}\,) \cdot (\vec{c} \times \vec{d}\,)$.

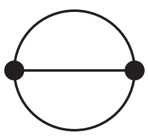
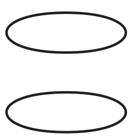
Exercise 2 *Show that* $\big((\vec{a} \times \vec{b}\,) \times \vec{c}\big) \times \vec{d} = \big((\vec{a} \times \vec{b}\,) \cdot \vec{d}\,\big)\vec{c} - (\vec{a} \times \vec{b}\,)(\vec{c} \cdot \vec{d}\,)$.

In birdtrack notation, it is immediately manifest that the triple product,

$(\vec{a} \times \vec{b}) \cdot \vec{c} =$  , (15)

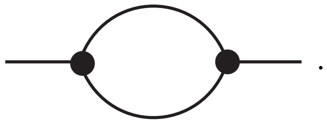
is invariant under cyclic permutations of the three vectors.

Taking Eq. (12) and joining the upper left to the upper right line and also the lower left to the lower right line yields

 =  −  $= 3 - 3 \cdot 3 = -6\,,$ (16)

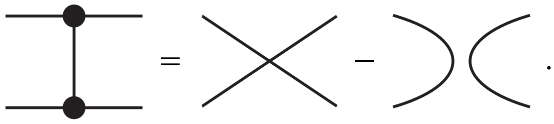
where we have used that each loop contributes a factor of $\delta_{jj} = 3$.

Exercise 3 *Evaluate*



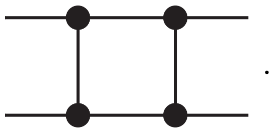
(17)

Rotating Eq. (12) by 90° we obtain the equivalent identity



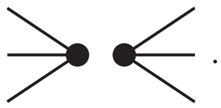
(18)

Exercise 4 *Evaluate*



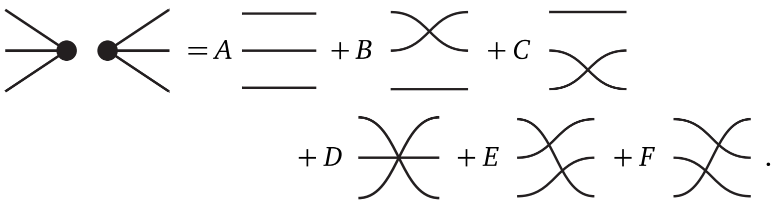
(19)

Finally, we want to study



(20)

Imagine for a moment assigning indices to the lines, then the diagram is non-zero only if the value of each index on the left matches the value of exactly one index on the right, i.e.



(21)

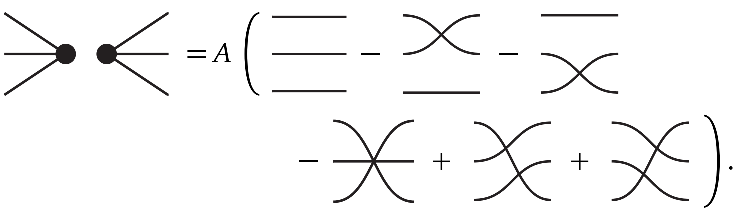
with some constants A to E. Intertwining, say the first two lines on the right, introduces a factor of (-1) on the l.h.s. of the equation, see eq. (6), whereas on the r.h.s. the roles of the terms interchange. Together this implies

$$B = -A\,, \quad F = -C \quad \text{and} \quad E = -D\,. \tag{22}$$

Similarly, by intertwining the lower two lines, we find

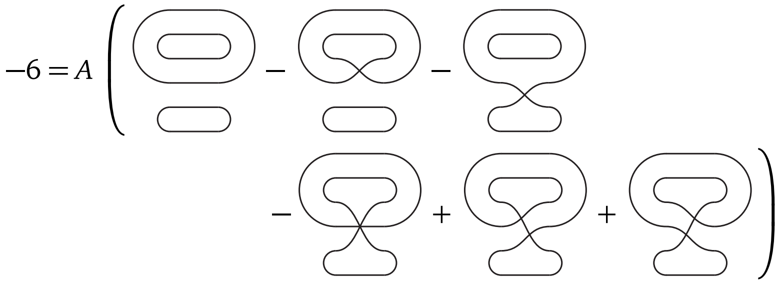
$$C = -A\,, \quad E = -B \quad \text{and} \quad F = -D\,, \tag{23}$$

and thus



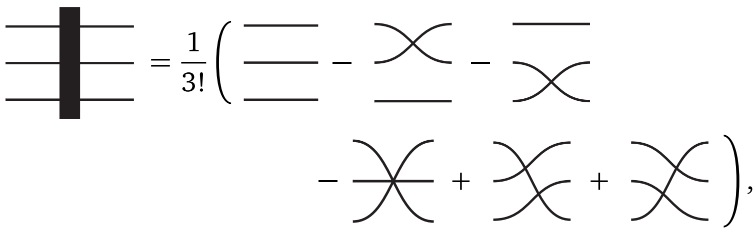
(24)

The factor A can, e.g., be determined by joining all lines on the left to those on the right (first to first etc.): On the l.h.s. we obtain −6, see Eq. (16), and hence



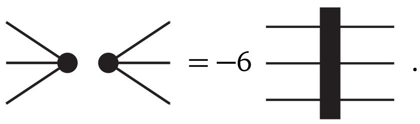
$$= A(27 - 9 - 9 - 9 + 3 + 3)$$
$$= 6A \quad \Leftrightarrow \quad A = -1. \tag{25}$$

Later, we will denote anti-symmetrisation of a couple of lines by a solid bar over these lines, normalised by the number of terms, i.e.



(26)

and likewise for symmetrisation using an open bar. With this notation we can rewrite our result as



(27)