Chapter Six

Permutations

The simplest example of invariant tensors is the products of Kronecker deltas. On tensor spaces they represent index permutations. This is the way in which the symmetric group S_p, the group of permutations of p objects, enters into the theory of tensor reps. In this chapter, I introduce birdtracks notation for permutations, symmetrizations and antisymmetrizations and collect a few results that will be useful later on. These are the (anti)symmetrization expansion formulas (6.10) and (6.19), Levi-Civita tensor relations (6.28) and (6.30), the characteristic equations (6.50), and the invariance conditions (6.54) and (6.56). The theory of Young tableaux (or plethysms) is developed in chapter 9.

6.1 SYMMETRIZATION

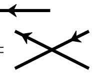
Operation of permuting tensor indices is a linear operation, and we can represent it by a $[d \times d]$ matrix:

$$\sigma_\alpha^\beta = \sigma_{b_1 \ldots b_p}^{a_1 a_2 \ldots a_q}{}_{,c_q \ldots c_2 c_1}^{d_p \ldots d_1} \,. \tag{6.1}$$

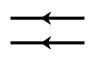
As the covariant and contravariant indices have to be permuted separately, it is sufficient to consider permutations of purely covariant tensors.

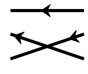
For 2-index tensors, there are two permutations:

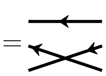
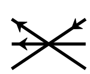
identity: $\mathbf{1}_{ab,}{}^{cd} = \delta_a^d \delta_b^c =$

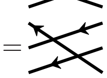
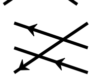
flip: $\sigma_{(12)ab,}{}^{cd} = \delta_a^c \delta_b^d =$  . (6.2)

For 3-index tensors, there are six permutations:

$\mathbf{1}_{a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \delta_{a_3}^{b_3} =$ 

$\sigma_{(12)a_1 a_2 a_3,}{}^{b_3 b_2 b_1} = \delta_{a_1}^{b_2} \delta_{a_2}^{b_1} \delta_{a_3}^{b_3} =$ 

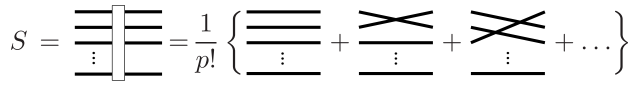
$\sigma_{(23)} =$  , $\sigma_{(13)} =$ 

$\sigma_{(123)} =$  , $\sigma_{(132)} =$  . (6.3)

Subscripts refer to the standard permutation cycles notation. For the remainder of this chapter we shall mostly omit the arrows on the Kronecker delta lines.

The symmetric sum of all permutations,

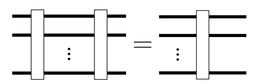
$$S_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_p}^{b_p} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_p}^{b_p} + \ldots\right\}$$



(6.4)

yields the symmetrization operator S. In birdtrack notation, a white bar drawn across p lines will always denote symmetrization of the lines crossed. A factor of $1/p!$ has been introduced in order for S to satisfy the projection operator normalization

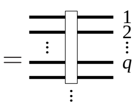
$$S^2 = S$$



(6.5)

A subset of indices $a_1, a_2, \ldots a_q$, $q < p$ can be symmetrized by symmetrization matrix $S_{12\ldots q}$

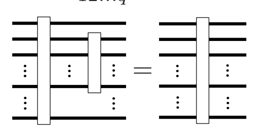
$$(S_{12\ldots q})_{a_1 a_2 \ldots a_q \ldots a_p},{}^{b_p \ldots b_q \ldots b_2 b_1} = \frac{1}{q!}\left\{\delta_{a_1}^{b_1}\delta_{a_2}^{b_2}\ldots\delta_{a_q}^{b_q} + \delta_{a_2}^{b_1}\delta_{a_1}^{b_2}\ldots\delta_{a_q}^{b_q} + \ldots\right\}\delta_{a_q+1}^{b_q+1}\ldots\delta_{a_p}^{b_p}$$

$S_{12\ldots q} =$ 

(6.6)

Overall symmetrization also symmetrizes any subset of indices:

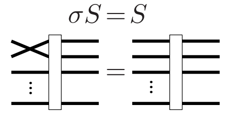
$$SS_{12\ldots q} = S$$



(6.7)

Any permutation has eigenvalue 1 on the symmetric tensor space:

$$\sigma S = S$$



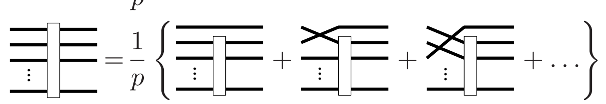
(6.8)

Diagrammatically this means that legs can be crossed and uncrossed at will.

The definition (6.4) of the symmetrization operator as the sum of all $p!$ permutations is inconvenient for explicit calculations; a recursive definition is more useful:

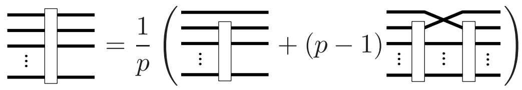
$$S_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = \frac{1}{p}\left\{\delta_{a_1}^{b_1} S_{a_2 \ldots a_p},{}^{b_p \ldots b_2} + \delta_{a_2}^{b_1} S_{a_1 a_3 \ldots a_p},{}^{b_p \ldots b_2} + \ldots\right\}$$

$$S = \frac{1}{p}\left(1 + \sigma_{(21)} + \sigma_{(321)} + \ldots + \sigma_{(p\ldots 321)}\right) S_{23\ldots p}$$

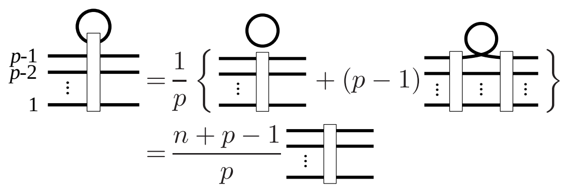


(6.9)

which involves only p terms. This equation says that if we start with the first index, we end up either with the first index, or the second index and so on. The remaining indices are fully symmetric. Multiplying by $S_{23} \ldots p$ from the left, we obtain an even more compact recursion relation with two terms only:

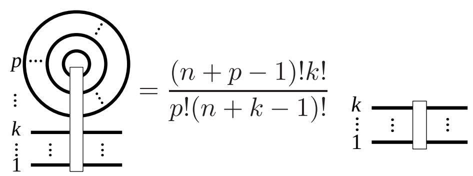

(6.10)

As a simple application, consider computation of a contraction of a single pair of indices:



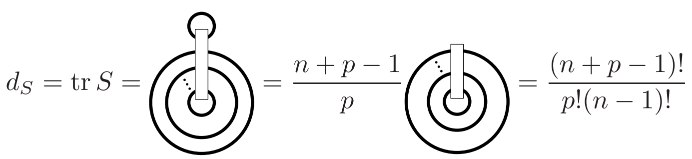
$$S_{a_p a_{p-1} \ldots a_1,}{}^{b_1 \ldots b_{p-1} a_p} = \frac{n+p-1}{p} S_{a_{p-1} \ldots a_1,}{}^{b_1 \ldots b_{p-1}} \,. \tag{6.11}$$

For a contraction in $(p-k)$ pairs of indices, we have



(6.12)

The trace of the symmetrization operator yields the number of independent components of fully symmetric tensors:



$$d_S = \mathrm{tr}\, S = \ldots = \frac{n+p-1}{p} \ldots = \frac{(n+p-1)!}{p!(n-1)!} \,. \tag{6.13}$$

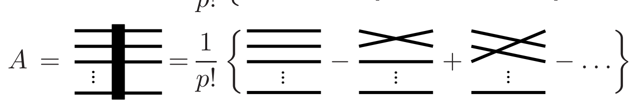
For example, for 2-index symmetric tensors,

$$d_S = n(n+1)/2 \,. \tag{6.14}$$

6.2 ANTISYMMETRIZATION

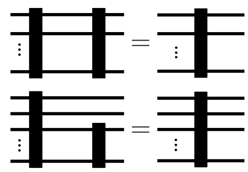
The alternating sum of all permutations,

$$A_{a_1 a_2 \ldots a_p,}{}^{b_p \ldots b_2 b_1} = \frac{1}{p!} \left\{ \delta_{a_1}^{b_1} \delta_{a_2}^{b_2} \ldots \delta_{a_p}^{b_p} - \delta_{a_2}^{b_1} \delta_{a_1}^{b_2} \ldots \delta_{a_p}^{b_p} + \ldots \right\}$$


(6.15)

yields the antisymmetrization projection operator A. In birdtrack notation, antisymmetrization of p lines will always be denoted by a black bar drawn across the lines. As in the previous section

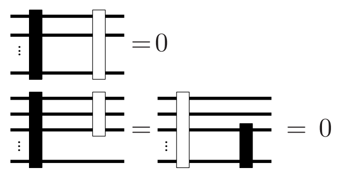
$$A^2 = A$$



(6.16)

and in addition

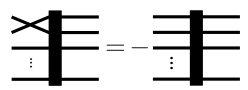
$$SA = 0$$



(6.17)

A transposition has eigenvalue -1 on the antisymmetric tensor space

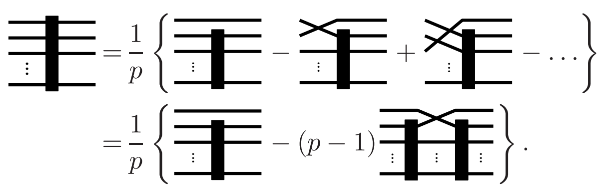
$$\sigma_{(i,i+1)} A = -A$$



(6.18)

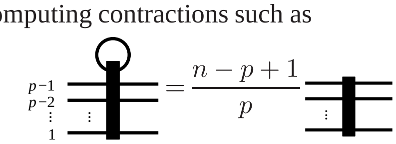
Diagrammatically this means that legs can be crossed and uncrossed at will, but with a factor of -1 for a transposition of any two neighboring legs.

As in the case of symmetrization operators, the recursive definition is often computationally convenient



(6.19)

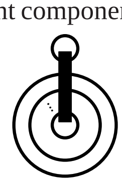
This is useful for computing contractions such as



$$A_{aa_{p-1}\ldots a_1,}{}^{b_1\ldots b_{p-1}a} = \frac{n-p+1}{p} A_{a_{p-1}\ldots a_1,}{}^{b_1\ldots b_{p-1}} \,. \tag{6.20}$$

The number of independent components of fully antisymmetric tensors is given by

$$d_A = \mathrm{tr}\, A =$$



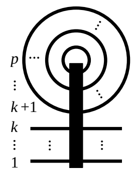
$$= \frac{n-p+1}{p}\frac{n-p+2}{p-1}\cdots\frac{n}{1}$$

$$= \begin{cases} \frac{n!}{p!(n-p)!}\,, & n \geq p \\ 0\,, & n < p \end{cases} \,. \tag{6.21}$$

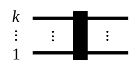
For example, for 2-index antisymmetric tensors the number of independent components is

$$d_A = \frac{n(n-1)}{2} \,. \tag{6.22}$$

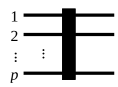
Tracing $(p-k)$ pairs of indices yields



$$= \frac{k!(n-k)!}{p!(n-p)!} \tag{6.23}$$



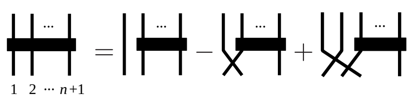
The antisymmetrization tensor $A_{a_1 a_2 \ldots ,}{}^{b_p \ldots b_2 b_1}$ has nonvanishing components, only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no nonvanishing components:



$$= 0 \qquad \text{if } p > n \,. \tag{6.24}$$

This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n+1$ case

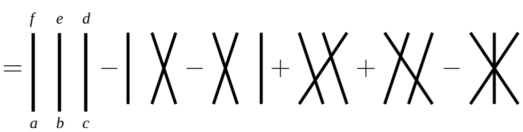
$0 =$



$$- \ldots \,. \tag{6.25}$$

For example, for $n = 2$ we have

$n = 2 :\quad 0 =$



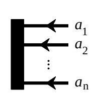
(6.26)

$$0 = \delta_a^f \delta_b^e \delta_c^d - \delta_a^f \delta_c^e \delta_b^d - \delta_b^f \delta_a^e \delta_c^d + \delta_b^f \delta_c^e \delta_a^d + \delta_c^f \delta_a^e \delta_b^d - \delta_c^f \delta_b^e \delta_a^d \,.$$

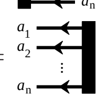
6.3 LEVI-CIVITA TENSOR

An antisymmetric tensor, with n indices in defining dimension n, has only one independent component ($d_n = 1$ by (6.21)). The clebsches (4.17) are in this case proportional to the *Levi-Civita tensor*:

$$(C_A)_1{}_{,}{}^{a_n \ldots a_2 a_1} = C \epsilon^{a_n \ldots a_2 a_1} =$$



$$(C_A)_{a_1 a_2 \ldots a_n}{}_{,}{}^{1} = C \epsilon_{a_1 a_2 \ldots a_n} =$$

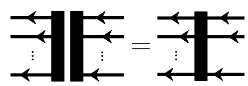


, (6.27)

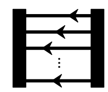
with $\epsilon^{12\ldots n} = \epsilon_{12\ldots n} = 1$. This diagrammatic notation for the Levi-Civita tensor was introduced by Penrose [281]. The normalization factors C are physically irrelevant.

They adjust the phase and the overall normalization in order that the Levi-Civita tensors satisfy the projection operator (4.18) and orthonormality (4.19) conditions:

$$\frac{1}{N!}\epsilon_{b_1 b_2 \dots b_n}\epsilon^{a_1 a_2 \dots a_n} = A_{b_1 b_2 \dots b_n},{}^{a_n \dots a_2 a_1}$$



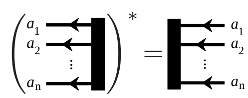
$$\frac{1}{N!}\epsilon_{a_1 a_2 \dots a_n}\epsilon^{a_1 a_2 \dots a_n} = \delta_{11} = 1\,, \qquad \text{[diagram]} = 1\,. \tag{6.28}$$



With our conventions,

$$C = \frac{i^{n(n-1)/2}}{\sqrt{n!}}. \tag{6.29}$$

The phase factor arises from the hermiticity condition (4.15) for clebsches (remember that indices are always read in the counterclockwise order around a diagram),



$$i^{-\phi}\epsilon_{a_1 a_2 \dots a_n} = i^{-\phi}\epsilon_{a_n \dots a_2 a_1}\,.$$

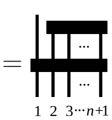
Transposing the indices

$$\epsilon_{a_1 a_2 \dots a_n} = -\epsilon_{a_2 a_1 \dots a_n} = \dots = (-1)^{n(n-1)/2}\epsilon_{a_n \dots a_2 a_1}\,,$$

yields $\phi = n(n-1)/2$. The factor $1/\sqrt{n!}$ is needed for the projection operator normalization (3.50).

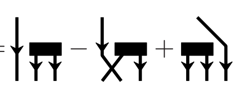
Given n dimensions we cannot label more than n indices, so Levi-Civita tensors satisfy

$$0 = \text{[diagram]}\,. \tag{6.30}$$



For example, for

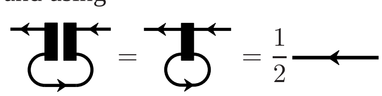
$$n = 2: \qquad 0 = \text{[diagram]}$$



$$0 = \delta_a^d\epsilon_{bc} - \delta_b^d\epsilon_{ac} + \delta_c^d\epsilon_{ab}\,. \tag{6.31}$$

This is actually the same as the completeness relation (6.28), as can be seen by contracting (6.31) with ϵ_{cd} and using

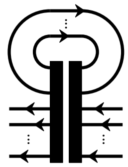
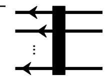
$$n = 2:$$



$$\epsilon_{ac}\epsilon^{bc} = \delta_a^b\,. \tag{6.32}$$

This relation is one of a series of relations obtained by contracting indices in the completeness relation (6.28) and substituting (6.23):

$$\epsilon_{a_n \dots a_{k+1} b_k \dots b_1}\epsilon^{a_n \dots a_{k+1} a_k \dots a_1} = k!(n-k)!\,A_{b_k \dots b_1},{}^{a_1 \dots a_k}$$

 $= \dfrac{k!(n-k)!}{n!}$  . (6.33)

Such identities are familiar from relativistic calculations $(n = 4)$:

$$
\begin{aligned}
\epsilon_{abcd}\epsilon^{agfe} &= \delta^{gfe}_{bcd}\,, & \epsilon_{abcd}\epsilon^{abfe} &= 2\delta^{fe}_{cd} \\
\epsilon_{abcd}\epsilon^{abce} &= 6\delta^{e}_{d}\,, & \epsilon_{abcd}\epsilon^{abcd} &= 24\,,
\end{aligned} \tag{6.34}
$$

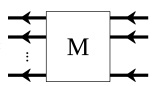
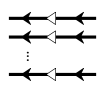
where the generalized Kronecker delta is defined by

$$
\frac{1}{p!}\delta^{b_1 b_2 \ldots b_p}_{a_1 a_2 \ldots a_p} = A_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} . \tag{6.35}
$$

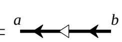
6.4 DETERMINANTS

Consider an $[n^p \times n^p]$ matrix $M_\alpha{}^\beta$ defined by a direct product of $[n \times n]$ matrices M_a^b

$$
M_\alpha{}^\beta = M_{a_1 a_2 \ldots a_p},{}^{b_p \ldots b_2 b_1} = M^{b_1}_{a_1} M^{b_2}_{a_2} \ldots M^{b_p}_{a_p}
$$

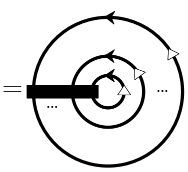
$M =$  $=$  , (6.36)

where

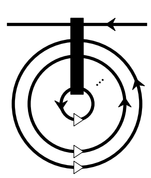
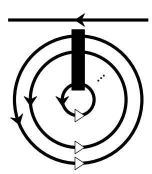
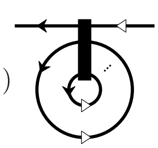
$M_a^b =$  . (6.37)

The trace of the antisymmetric projection of $M_\alpha{}^\beta$ is given by

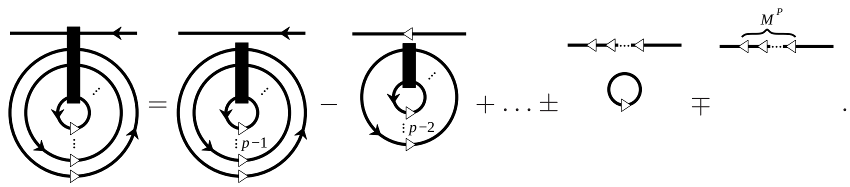
$$
\mathrm{tr}_p\, AM = A_{abc\ldots d},{}^{d'\ldots c' b' a'}\, M^a_{a'} M^b_{b'} \ldots M^d_{d'}
$$

$=$  . (6.38)

The subscript p on $\mathrm{tr}_p(\ldots)$ distinguishes the traces on $[n^p \times n^p]$ matrices M_α^β from the $[n \times n]$ matrix trace $\mathrm{tr}\, M$. To derive a recursive evaluation rule for $\mathrm{tr}_p\, AM$, use (6.19) to obtain

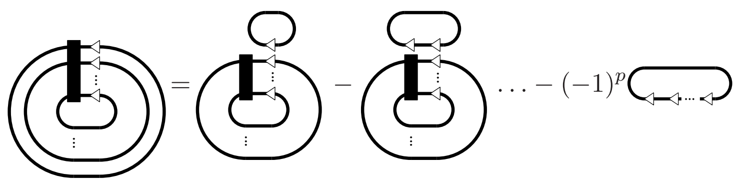
 $= \dfrac{1}{p}\Big\{$  $- (p-1)$  $\Big\}$. (6.39)

Iteration yields



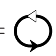
(6.40)

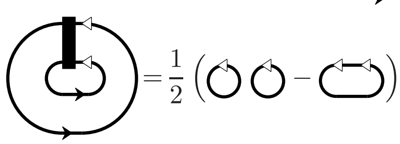
Contracting with M_a^b, we obtain



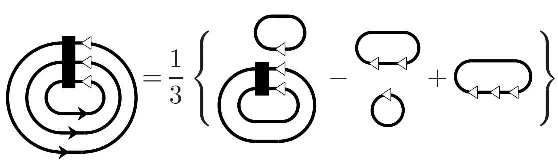
$$\mathrm{tr}_p\, AM = \frac{1}{p}\sum_{k=1}^{p}(-1)^{k-1}\,(\mathrm{tr}_{p-k}\, AM)\,\mathrm{tr}\, M^k\,. \tag{6.41}$$

This formula enables us to compute recursively all $\mathrm{tr}_p\, AM$ as polynomials in traces of powers of M:

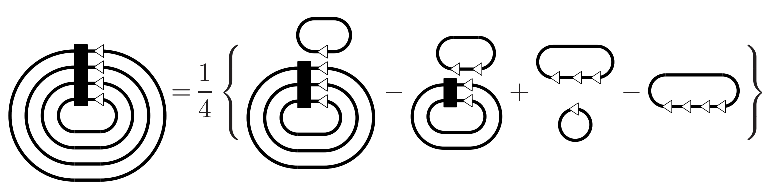
$\mathrm{tr}_0\, AM = 1\,, \qquad \mathrm{tr}_1\, AM =$  $= \mathrm{tr}\, M$ (6.42)



$$\mathrm{tr}_2\, AM = \frac{1}{2}\left\{(\mathrm{tr}\, M)^2 - \mathrm{tr}\, M^2\right\} \tag{6.43}$$



$$\mathrm{tr}_3\, AM = \frac{1}{3!}\left\{(\mathrm{tr}\, M)^3 - 3(\mathrm{tr}\, M)(\mathrm{tr}\, M^2) + 2\ \mathrm{tr}\, M^3\right\} \tag{6.44}$$



$$\begin{aligned}\mathrm{tr}_4\, AM = \frac{1}{4!}\big\{&(\mathrm{tr}\, M)^4 - 6(\mathrm{tr}\, M)^2\,\mathrm{tr}\, M^2 \\ &+ 3(\mathrm{tr}\, M^2)^2 + 8\ \mathrm{tr}\, M^3\,\mathrm{tr}\, M - 6\ \mathrm{tr}\, M^4\big\}\,.\end{aligned} \tag{6.45}$$

For $p = n$ (M_a^b are $[n \times n]$ matrices) the antisymmetrized trace is the determinant

$$\det M = \mathrm{tr}_n\, AM = A_{a_1 a_2 \ldots a_n},{}^{b_n \ldots b_2 b_1}\, M_{b_1}^{a_1} M_{b_2}^{a_2} \ldots M_{b_n}^{a_n} . \qquad (6.46)$$

The coefficients in the above expansions are simple combinatoric numbers. A general term for $(\mathrm{tr}\, M^{\ell_1})^{\alpha_1} \cdots (\mathrm{tr}\, M^{\ell_s})^{\alpha_s}$, with α_1 loops of length ℓ_1, α_2 loops of length ℓ_2 and so on, is divided by the number of ways in which this pattern may be obtained:

$$\ell_1^{\alpha_1} \ell_2^{\alpha_2} \ldots \ell_s^{\alpha_s} \alpha_1! \alpha_2! \ldots \alpha_s! . \qquad (6.47)$$

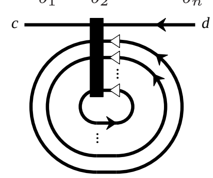
6.5 CHARACTERISTIC EQUATIONS

We have noted that the dimension of the antisymmetric tensor space is zero for $n < p$. This is rather obvious; antisymmetrization allows each label to be used at most once, and it is impossible to label more legs than there are labels. In terms of the antisymmetrization operator this is given by the identity

$$A = 0 \qquad \text{if } \; p > n . \qquad (6.48)$$

This trivial identity has an important consequence: it guarantees that any $[n \times n]$ matrix satisfies a characteristic (or Hamilton-Cayley or secular) equation. Take $p = n + 1$ and contract with M_a^b n index pairs of A:

$$A_{c a_1 a_2 \ldots a_n},{}^{b_n \ldots b_2 b_1 d}\, M_{b_1}^{a_1} M_{b_2}^{a_2} \ldots M_{b_n}^{a_n} = 0$$



$$= 0 . \qquad (6.49)$$

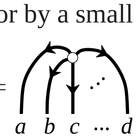
We have already expanded this in (6.40). For $p = n+1$ we obtain the *characteristic equation*

$$0 = \sum_{k=0}^{n} (-1)^k (\mathrm{tr}_{n-k}\, AM) M^k , \qquad (6.50)$$
$$= M^n - (\mathrm{tr}\, M) M^{n-1} + (\mathrm{tr}_2\, AM)\, M^{n-2} - \ldots + (-1)^n (\det M)\, \mathbf{1} .$$

6.6 FULLY (ANTI)SYMMETRIC TENSORS

We shall denote a fully *symmetric* tensor by a small circle (white dot)

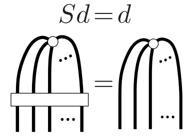
$$d_{abc \ldots f} = $$



$$. \qquad (6.51)$$

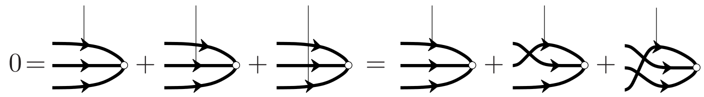
A symmetric tensor $d_{abc \ldots d} = d_{bac \ldots d} = d_{acb \ldots d} = \ldots$ satisfies

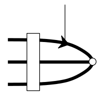
$$Sd = d$$



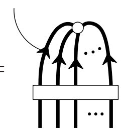
$$. \qquad (6.52)$$

If this tensor is also an invariant tensor, the invariance condition (4.36) can be written as

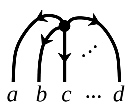


$= p$  $\quad (p = \text{number of indices})\,.$ (6.53)

Hence, the invariance condition for symmetric tensors is

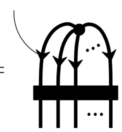
$0 =$  . (6.54)

The fully *antisymmetric* tensors with *odd* numbers of legs will be denoted by black dots

$f_{abc\ldots d} =$ 

, (6.55)

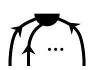
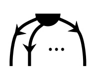
with the invariance condition stated compactly as

$0 =$  . (6.56)

If the number of legs is *even*, an antisymmetric tensor is *anticyclic*,

$$f_{abc\ldots d} = -f_{bc\ldots da}\,, \tag{6.57}$$

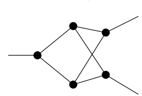
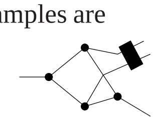
and the birdtrack notation must distinguish the first leg. A black dot is inadequate for the purpose. A bar, as for the Levi-Civita tensor (6.27), or a semicircle for the symplectic invariant introduced below in (12.3), and fully skew-symmetric invariant tensors investigated in (15.27)

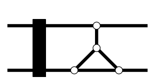
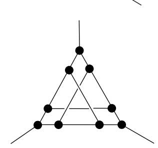
$f^{ab\ldots c} =$  , $\qquad f_{ab\ldots c} =$  (6.58)

or a similar notation fixes the problem.

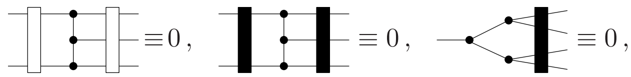
6.7 IDENTICALLY VANISHING TENSORS

Noting that a given group-theoretic weight vanishes *identically* is often an important step in a birdtrack calculation. Some examples are

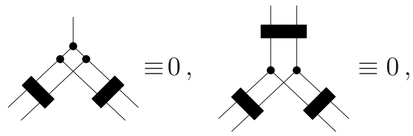
 $\equiv 0\,,$  $\equiv 0\,,$ (6.59)

 $\equiv 0\,,$  $\equiv 0\,.$ (6.60)

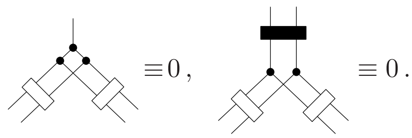
In graph theory [268, 294] the left graph in (6.59) is known as the Kuratowsky graph, and the right graph in (6.60) as the Peterson graph.



(6.61)



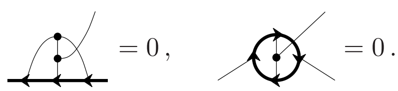
(6.62)



(6.63)

The above identities hold for any antisymmetric 3-index tensor; in particular, they hold for the Lie algebra structure constants iC_{ijk}. They are proven by mapping a diagram into itself by index transpositions. For example, interchange of the top and bottom vertices in (6.59) maps the diagram into itself, but with the $(-1)^5$ factor.

From the Lie algebra (4.47) it also follows that for any irreducible rep we have



(6.64)