8.2 CHARACTERS

Physics calculations (such as lattice gauge theories) often involve group-invariant quantities formed by contracting G with invariant tensors. Such invariants are of the form $\mathrm{tr}(hG) = h_b{}^a G_a{}^b$, where h stands for any invariant tensor. The trace of an irreducible $[d \times d]$ matrix rep λ of g is called the *character* of the rep:

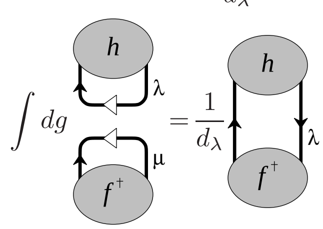
$$\chi_\lambda(g) = \mathrm{tr}\, G^{(\lambda)} = G^{(\lambda)}{}_a{}^a \,. \tag{8.24}$$

The character of the conjugate rep is

$$\chi^\lambda(g) = \mathrm{tr}\, G^{(\lambda)\dagger} = G^{(\lambda)\,a}{}_a = \chi_\lambda(g)^* \,. \tag{8.25}$$

Contracting (8.14) with two arbitrary invariant $[d \times d]$ tensors $h_d{}^a$ and $(f^\dagger)_b{}^c$, we obtain the *character orthonormality relation*:

$$\int dg\, \chi_\lambda(hg)\chi^\mu(gf) = \delta^\mu_\lambda \frac{1}{d_\lambda} \chi_\lambda(hf^\dagger) \tag{8.26}$$



$\left(\begin{array}{l} \lambda, \mu \text{ irreducible} \\ \text{reps} \end{array} \right)$.

The character orthonormality tells us that if two group-invariant quantities share a $GG^\dagger$ pair, the group averaging sews them into a single group-invariant quantity. The replacement of $G_a{}^b$ by the character $\chi_\lambda(h^\dagger g)$ does not mean that any of the tensor index structure is lost; $G_a{}^b$ can be recovered by differentiating

$$G_a{}^b = \frac{d}{dh_b{}^a} \chi_\lambda(h^\dagger g) \,. \tag{8.27}$$

The birdtracks and the characters are two equivalent notations for evaluating group integrals.