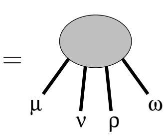
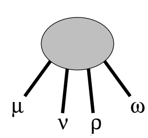
5.3 WIGNER-ECKART THEOREM

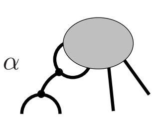
For concreteness, consider an arbitrary invariant tensor with four indices:

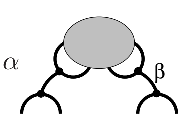
$$T = $$ 

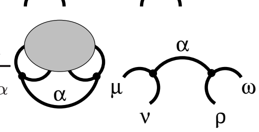
$$, \qquad (5.19)$$

where μ, ν, ρ and ω are rep labels, and indices and line arrows are suppressed. Now insert repeatedly the completeness relation (5.8) to obtain



$$= \sum_{\alpha} \frac{1}{a_\alpha} \alpha$$ 

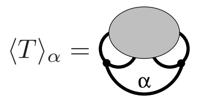
$$= \sum_{\alpha,\beta} \frac{1}{a_\alpha a_\beta} \alpha$$ 

$$= \sum_{\alpha} \frac{1}{a_\alpha^2} \frac{1}{d_\alpha}$$ 

$$. \qquad (5.20)$$

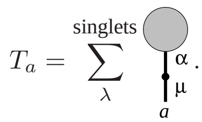
In the last line we have used the orthonormality of projection operators — as in (5.13) or (5.23).

In this way any invariant tensor can be reduced to a sum over clebsches (*kinematics*) weighted by *reduced matrix elements*:



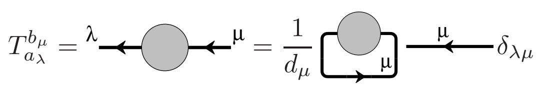
(5.21)

This theorem has many names, depending on how the indices are grouped. If T is a vector, then only the 1-dimensional reps (singlets) contribute



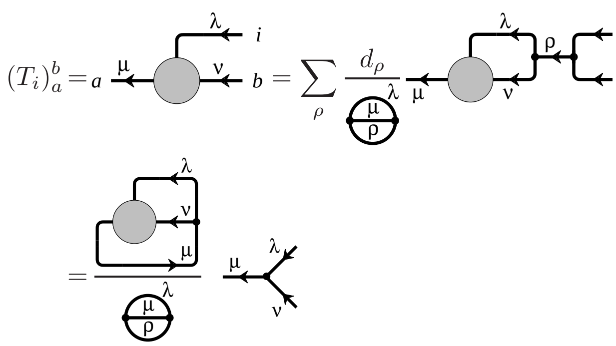
(5.22)

If T is a matrix, and the reps α, μ are irreducible, the theorem is called *Schur's Lemma* (for an irreducible rep an invariant matrix is either zero, or proportional to the unit matrix):



(5.23)

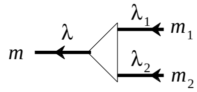
If T is an "invariant tensor operator," then the theorem is called the *Wigner-Eckart theorem* [347, 107]:



(5.24)

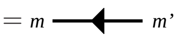
(assuming that μ appears only once in $\overline{\lambda} \otimes \mu$ Kronecker product). If T has many indices, as in our original example (5.19), the theorem is ascribed to Yutsis, Levinson, and Vanagas [359]. The content of all these theorems is that they reduce spectroscopic calculations to evaluation of "vacuum bubbles" or "reduced matrix elements" (5.21).

The rectangular matrices $(C_\lambda)_\sigma^\alpha$ from (3.27) do not look very much like the clebsches from the quantum mechanics textbooks; neither does the Wigner-Eckart theorem in its birdtrack version (5.24). The difference is merely a difference of notation. In the bra-ket formalism, a clebsch for $\lambda_1 \otimes \lambda_2 \to \lambda$ is written as

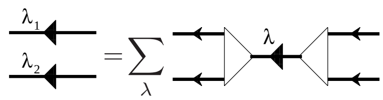


$$= \langle \lambda_1 \lambda_2 \lambda m | \lambda_1 m_1 \lambda_2 m_2 \rangle \,. \quad (5.25)$$

Representing the $[d_\lambda \times d_\lambda]$ rep of a group element g diagrammatically by a black triangle,

$$D^{\lambda}_{m,m'}(g) = $$  , (5.26)

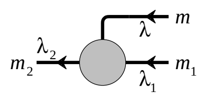
we can write the Clebsch-Gordan series (3.49) as



$$D^{\lambda_1}_{m_1 m'_1}(g) D^{\lambda_2}_{m_2 m'_2}(g) =$$

$$\sum_{\tilde\lambda, \tilde m, \tilde m_1} \langle \lambda_1 m_1 \lambda_2 m_2 | \lambda_1 \lambda_2 \tilde\lambda \tilde m \rangle D^{\tilde\lambda}_{\tilde m \tilde m_1}(g) \langle \lambda_1 \lambda_2 \tilde\lambda \tilde m_1 | \lambda_1 m'_1 \lambda_2 m'_2 \rangle \,.$$

An "invariant tensor operator" can be written as

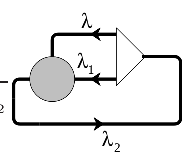
$$\langle \lambda_2 m_2 | T^{\lambda}_{m} | \lambda_1 m_1 \rangle \; = $$  . (5.27)

In the bra-ket formalism, the Wigner-Eckart theorem (5.24) is written as

$$\langle \lambda_2 m_2 | T^{\lambda}_{m} | \lambda_1 m_1 \rangle = \langle \lambda \lambda_1 \lambda_2 m_2 | \lambda m \lambda_1 m_1 \rangle T(\lambda, \lambda_1 \lambda_2) \,, \tag{5.28}$$

where the reduced matrix element is given by

$$T(\lambda, \lambda_1 \lambda_2) = \frac{1}{d_{\lambda_2}} \sum_{n_1, n_2, n} \langle \lambda n \lambda_1 n_1 | \lambda \lambda_1 \lambda_2 n_2 \rangle \langle \lambda_2 n_2 | T^{\lambda}_{n} | \lambda_1 n_1 \rangle$$

$$= \frac{1}{d_{\lambda_2}}$$  . (5.29)

We do not find the bra-ket formalism convenient for the group-theoretic calculations that will be discussed here.