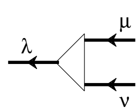
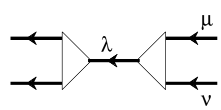
Clebsches project a tensor in $V^p \otimes \bar{V}^q$ onto a subspace $\lambda$.
In practice one usually reduces a tensor step by step, decomposing a 2-particle state at each step. While there is some arbitrariness in the order in which these reductions are carried out, the final result is invariant and highly elegant: any group-theoretical invariant quantity can be expressed in terms of Wigner 3- and 6-$j$ coefficients.

## 5.1 COUPLINGS AND RECOUPLINGS

We denote the clebsches for $\mu \otimes \nu \to \lambda$ by



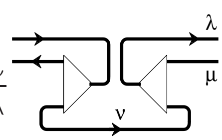

, $\mathbf{P}_\lambda =$




. (5.1)

Here $\lambda, \mu, \nu$ are rep labels, and the corresponding tensor indices are suppressed. Furthermore, if $\mu$ and $\nu$ are irreducible reps, the same clebsches can be used to project $\mu \otimes \bar{\lambda} \to \bar{\nu}$

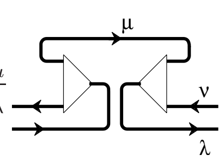
$\mathbf{P}_\nu = \frac{d_\nu}{d_\lambda}$




, (5.2)

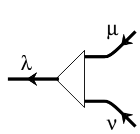
and $\nu \otimes \bar{\lambda} \to \bar{\mu}$

$\mathbf{P}_\mu = \frac{d_\mu}{d_\lambda}$

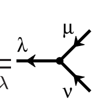



. (5.3)

Here the normalization factors come from $P^2 = P$ condition. In order to draw the projection operators in a more symmetric way, we replace clebsches by 3-vertices:

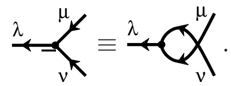



$\equiv \frac{1}{\sqrt{a_\lambda}}$



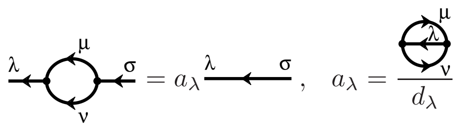

. (5.4)

In this definition one has to keep track of the ordering of the lines around the vertex. If in some context the birdtracks look better with two legs interchanged, one can use Yutsis's notation [359]:



(5.5)

While all sensible clebsches are normalized by the orthonormality relation (4.19), in practice no two authors ever use the same normalization for 3-vertices (in other guises known as 3-$j$ coefficients, Gell-Mann $\lambda$ matrices, Cartan roots, Dirac $\gamma$ matrices, *etc.*). For this reason we shall usually not fix the normalization

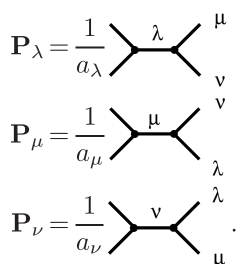


(5.6)

leaving the reader the option of substituting his or her favorite choice (such as $a = \frac{1}{2}$ if the 3-vertex stands for Gell-Mann $\frac{1}{2}\lambda_i$, *etc.*).

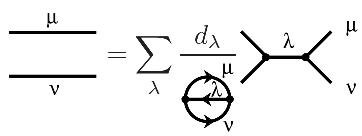
To streamline the discussion, we shall drop the arrows and most of the rep labels in the remainder of this chapter — they can always easily be reinstated.

The above three projection operators now take a more symmetric form:



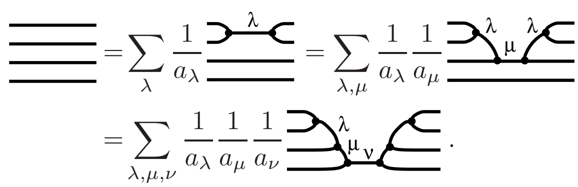
(5.7)

In terms of 3-vertices, the completeness relation (4.20) is



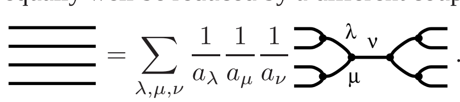
(5.8)

Any tensor can be decomposed by successive applications of the completeness relation:



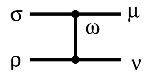
(5.9)

Hence, if we know clebsches for $\lambda \otimes \mu \to \nu$, we can also construct clebsches for $\lambda \otimes \mu \otimes \nu \otimes \ldots \to \rho$. However, there is no unique way of building up the clebsches; the above state can equally well be reduced by a different coupling scheme

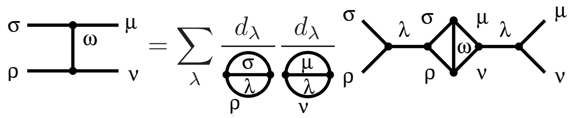


(5.10)

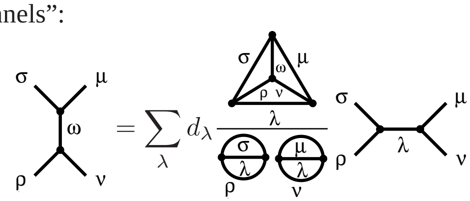
Consider now a process in which a particle in the rep $\mu$ interacts with a particle in the rep $\nu$ by exchanging a particle in the rep $\omega$:

 . (5.11)

The final particles are in reps $\rho$ and $\sigma$. To evaluate the contribution of this exchange to the spectroscopic levels of the $\mu$-$\nu$ particles system, we insert the Clebsch-Gordan series (5.8) twice, and eliminate one of the sums by the orthonormality relation (5.6):

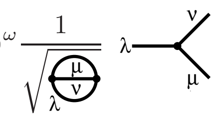
 . (5.12)

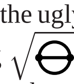
By assumption $\lambda$ is an irrep, so we have a recoupling relation between the exchanges in "$s$" and "$t$ channels":

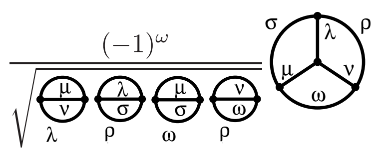
 . (5.13)

We shall refer to  as 3-$j$ coefficients and  as 6-$j$ coefficients, and commit ourselves to no particular normalization convention.

In atomic physics it is customary to absorb  into the 3-vertex and define a 3-$j$ symbol [238, 287, 347]

$$\begin{pmatrix} \lambda & \mu & \nu \\ \alpha & \beta & \gamma \end{pmatrix} = (-1)^{\omega}$$  . (5.14)

Here $\alpha = 1, 2, \ldots, d_\lambda$, *etc.*, are indices, $\lambda, \mu, \nu$ rep labels and $\omega$ the phase convention. Fixing a phase convention is a waste of time, as the phases cancel in summed-over quantities. All the ugly square roots, one remembers from quantum mechanics, come from sticking  into 3-$j$ symbols. Wigner [347] 6-$j$ *symbols* are related to our 6-$j$ *coefficients* by

$$\begin{Bmatrix} \lambda & \mu & \nu \\ \omega & \rho & \sigma \end{Bmatrix} =$$  . (5.15)

The name $3n$-$j$ symbol comes from atomic physics, where a recoupling involves $3n$ angular momenta $j_1, j_2, \ldots, j_{3n}$ (see section 14.2).

Most of the textbook symmetries of and relations between 6-$j$ symbols are obvious from looking at the corresponding diagrams; others follow quickly from completeness relations.

If we know the necessary 6-$j$'s, we can compute the level splittings due to single particle exchanges. In the next section we shall show that a far stronger claim can be made: given the 3- and 6-$j$ coefficients, we can compute *all* multiparticle matrix elements.

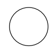
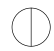
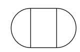
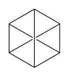
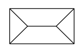
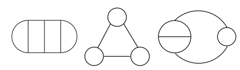
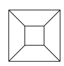
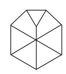
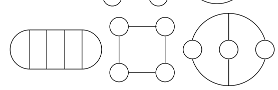
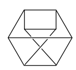
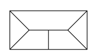
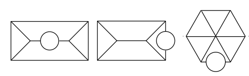
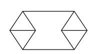
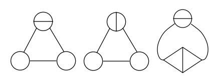
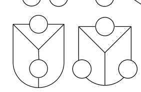
| | Skeletons | Vertex insertions | Self-energy insertions | Total number |
|---|---|---|---|---|
| 1-$j$ |  | | | 1 |
| 3$j$ | | |  | 1 |
| 6-$j$ | |  |  | 2 |
| 9-$j$ |  |  |  | 5 |
| 12-$j$ |  |  |  | 16 |
| |  |  |  | |
| | |  |  | |
| | | |  | |

Table 5.1 Topologically distinct types of Wigner $3n$-$j$ coefficients, enumerated by drawing all possible graphs, eliminating the topologically equivalent ones by hand. Lines meeting in any 3-vertex correspond to any three irreducible representations with a nonvanishing Clebsch-Gordan coefficient, so in general these graphs cannot be reduced to simpler graphs by means of such as the Lie algebra (4.47) and Jacobi identity (4.48).