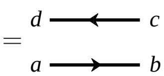
2.2 FIRST EXAMPLE: $SU(n)$

How do we describe the invariance group that preserves the norm of a complex vector? The *list of primitives* consists of a single primitive invariant,

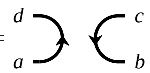
$$m(p,q) = \delta_b^a p^b q_a = \sum_{a=1}^{n} (p_a)^* q_a \,.$$

The Kronecker δ_b^a is the only primitive invariant tensor. We can immediately write down the two *invariant matrices* on the tensor product of the defining space and its conjugate,

$$\text{identity} : \mathbf{1}_{d,b}^{a\,c} = \delta_b^a \delta_d^c =$$



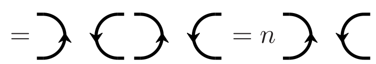
$$\text{trace} : T_{d,b}^{a\,c} = \delta_d^a \delta_b^c =$$



The *characteristic equation* for T written out in the matrix, tensor, and birdtrack notations is

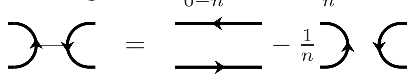
$$T^2 = nT$$

$$T_{d,e}^{a\,f} T_{f,b}^{e\,c} = \delta_d^a \delta_e^f \delta_f^e \delta_b^c = n\, T_{d,b}^{a\,c}$$



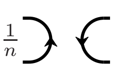
Here we have used $\delta_e^e = n$, the dimension of the defining vector space. The roots are $\lambda_1 = 0$, $\lambda_2 = n$, and the corresponding *projection operators* are

$$SU(n) \text{ adjoint rep:} \qquad \mathbf{P}_1 = \frac{T - n\mathbf{1}}{0-n} = \mathbf{1} - \frac{1}{n}T$$



(2.5)

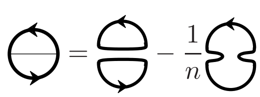
$$U(n) \text{ singlet:} \qquad \mathbf{P}_2 = \frac{T - 0\cdot\mathbf{1}}{n-0} = \frac{1}{n}T = \frac{1}{n}$$



Now we can evaluate any number associated with the $SU(n)$ adjoint rep, such as its dimension and various casimirs.

The *dimensions* of the two reps are computed by tracing the corresponding projection operators (see section 3.5):

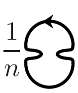
$$SU(n) \text{ adjoint:} \quad d_1 = \operatorname{tr} \mathbf{P}_1 =$$



$$= \delta_b^b \delta_a^a - \frac{1}{n} \delta_a^b \delta_b^a$$

$$= n^2 - 1$$

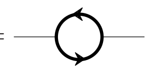
$$\text{singlet:} \quad d_2 = \operatorname{tr} \mathbf{P}_2 = \frac{1}{n}$$



$$= 1 \,.$$

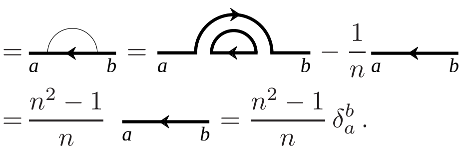
To evaluate *casimirs*, we need to fix the overall normalization of the generators T_i of $SU(n)$. Our convention is to take

$$\delta_{ij} = \mathrm{tr}\, T_i T_j =$$



The value of the quadratic casimir for the defining rep is computed by substituting the adjoint projection operator:

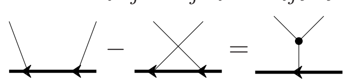
$$SU(n): \quad C_F \delta_a^b = (T_i T_i)_a^b =$$



$$= \frac{n^2-1}{n}\,\delta_a^b\,. \qquad (2.6)$$

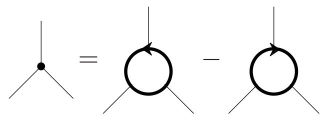
In order to evaluate the quadratic casimir for the adjoint rep, we need to replace the structure constants iC_{ijk} by their *Lie algebra* definition (see section 4.5)

$$T_i T_j - T_j T_i = i C_{ij\ell} T_\ell$$



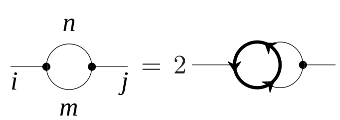
Tracing with T_k, we can express C_{ijk} in terms of the defining rep traces:

$$iC_{ijk} = \mathrm{tr}(T_i T_j T_k) \; - \; \mathrm{tr}(T_j T_i T_k)$$

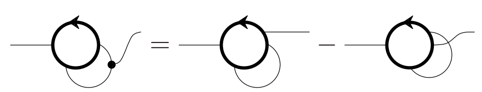


The adjoint quadratic casimir $C_{imn}C^{nmj}$ is now evaluated by first eliminating C_{ijk}'s in favor of the defining rep:

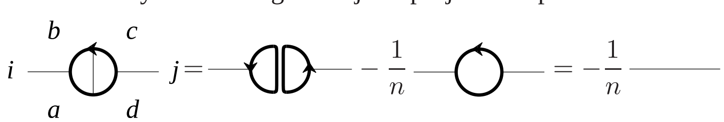
$$\delta_{ij} C_A =$$



The remaining C_{ijk} can be unwound by the Lie algebra commutator:



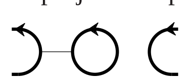
We have already evaluated the quadratic casimir (2.6) in the first term. The second term we evaluate by substituting the adjoint projection operator



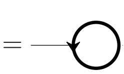
$$\mathrm{tr}(T_i T_k T_j T_k) = (T_i)_a^b (P_1)_{d,}^{a}{}_{b}^{c} (T_j)_c^d = (T_i)_a^a (T_j)_c^c - \frac{1}{n}(T_i)_a^b (T_j)_b^a\,.$$

The $(T_i)_a^a (T_j)_c^c$ term vanishes by the tracelessness of T_i's. This is a consequence of the orthonormality of the two projection operators $\mathbf{P}_1$ and $\mathbf{P}_2$ in (2.5) (see (3.50)):

$$0 = \mathbf{P}_1 \mathbf{P}_2 =$$



$$\Rightarrow \mathrm{tr}\, T_i =$$

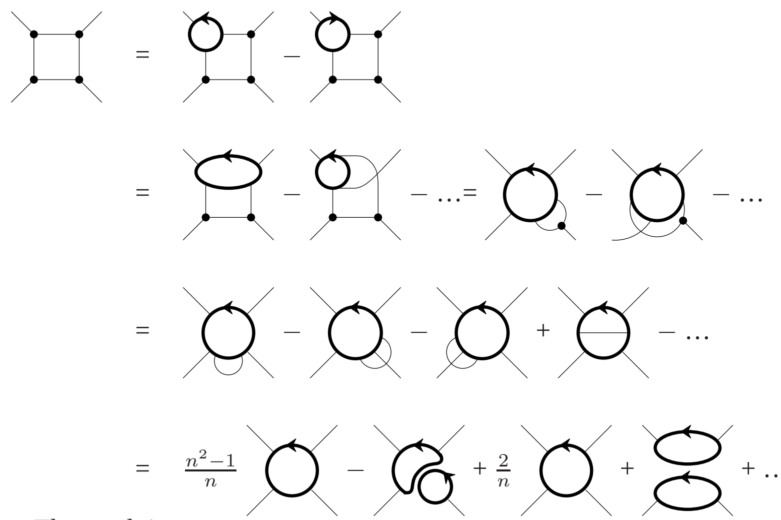


$$= 0\,.$$

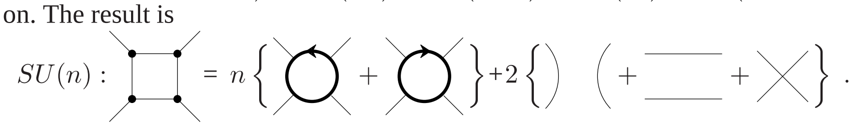
Combining the above expressions we finally obtain

$$C_A = 2\left(\frac{n^2-1}{n} + \frac{1}{n}\right) = 2n\ .$$

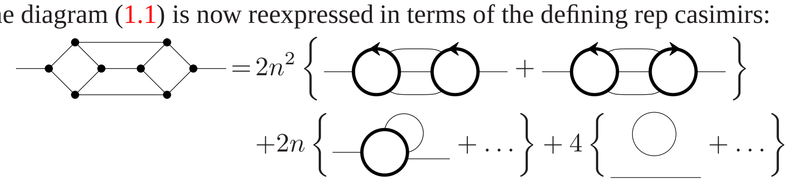
The problem (1.1) that started all this is evaluated the same way. First we relate the adjoint quartic casimir to the defining casimirs:



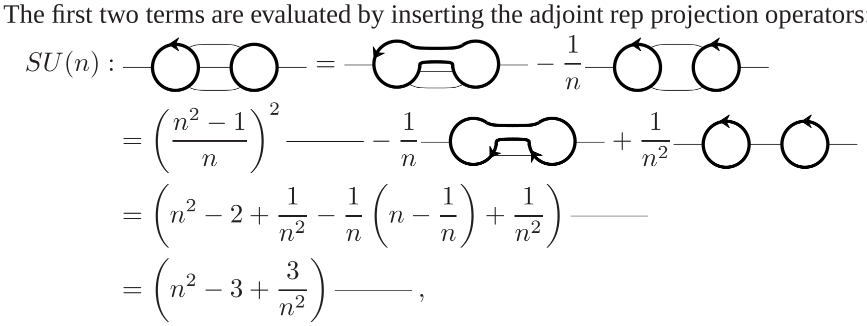
and so on. The result is



The diagram (1.1) is now reexpressed in terms of the defining rep casimirs:



The first two terms are evaluated by inserting the adjoint rep projection operators:



and the remaining terms have already been evaluated. Collecting everything together, we finally obtain