Chapter Nine

Unitary groups

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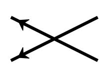
$U(n)$ is the group of all transformations that leave invariant the norm $\overline{q}q = \delta_b^a q^b q_a$ of a complex vector q. For $U(n)$ there are no other invariant tensors beyond those constructed of products of Kronecker deltas. They can be used to decompose the tensor reps of $U(n)$. For purely covariant or contravariant tensors, the symmetric group can be used to construct the Young projection operators. In sections. 9.1–9.2 we show how to do this for 2- and 3-index tensors by constructing the appropriate characteristic equations.

For tensors with more indices it is easier to construct the Young projection operators directly from the Young tableaux. In section 9.3 we review the Young tableaux, and in section 9.4 we show how to construct Young projection operators for tensors with any number of indices. As examples, 3- and 4-index tensors are decomposed in section 9.5. We use the projection operators to evaluate $3n$-j coefficients and characters of $U(n)$ in sections. 9.6–9.9, and we derive new sum rules for $U(n)$ 3-j and 6-j symbols in section 9.7. In section 9.8 we consider the consequences of the Levi-Civita tensor being an extra invariant for $SU(n)$.

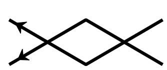
For mixed tensors the reduction also involves index contractions and the symmetric group methods alone do not suffice. In sections. 9.10–9.12 the mixed $SU(n)$ tensors are decomposed by the projection operator techniques introduced in chapter 3. $SU(2)$, $SU(3)$, $SU(4)$, and $SU(n)$ are discussed from the “invariance group" perspective in chapter 15.

9.1 TWO-INDEX TENSORS

Consider 2-index tensors $q^{(1)} \otimes q^{(2)} \in \otimes V^2$. According to (6.1), all permutations are represented by invariant matrices. Here there are only two permutations, the identity and the flip (6.2),

$\sigma =$  .

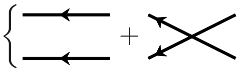
The flip satisfies

$\sigma^2 =$  $= \mathbf{1}\,,$

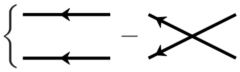
$$(\sigma + \mathbf{1})(\sigma - \mathbf{1}) = 0\,. \qquad (9.1)$$

The eigenvalues are $\lambda_1 = 1, \lambda_2 = -1$, and the corresponding projection operators (3.48) are

$$\mathbf{P}_1 = \frac{\sigma - (-1)\mathbf{1}}{1-(-1)} = \frac{1}{2}(\mathbf{1}+\sigma) = \frac{1}{2}\left\{ \quad \right\}, \tag{9.2}$$

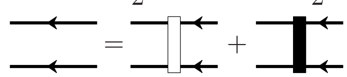


$$\mathbf{P}_2 = \frac{\sigma - \mathbf{1}}{-1-1} = \frac{1}{2}(\mathbf{1}-\sigma) = \frac{1}{2}\left\{ \quad \right\}. \tag{9.3}$$



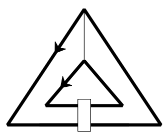
We recognize the symmetrization, antisymmetrization operators (6.4), (6.15); $\mathbf{P}_1 = \mathbf{S}$, $\mathbf{P}_2 = \mathbf{A}$, with subspace dimensions $d_1 = n(n+1)/2, d_2 = n(n-1)/2$. In other words, under general linear transformations the symmetric and the antisymmetric parts of a tensor x_{ab} transform separately:

$$x = \mathbf{S}x + \mathbf{A}x\,,$$
$$x_{ab} = \frac{1}{2}(x_{ab}+x_{ba}) + \frac{1}{2}(x_{ab}-x_{ba})$$

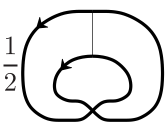


(9.4)

The Dynkin indices for the two reps follow by (7.29) from $6j's$:



$$= \frac{1}{2}(0) + \frac{1}{2} \quad = \frac{N}{2}$$



$$\ell_1 = \frac{2\ell}{n}\cdot d_1 + \frac{2\ell}{N}\cdot\frac{N}{2}$$
$$= \ell(n+2)\,. \tag{9.5}$$

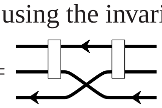
Substituting the defining rep Dynkin index $\ell^{-1} = C_A = 2n$, computed in section 2.2, we obtain the two Dynkin indices

$$\ell_1 = \frac{n+2}{2n}\,, \qquad \ell_2 = \frac{n-2}{2n}\,. \tag{9.6}$$

9.2 THREE-INDEX TENSORS

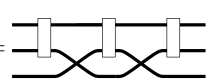
Three-index tensors can be reduced to irreducible subspaces by adding the third index to each of the 2-index subspaces, the symmetric and the antisymmetric. The results of this section are summarized in figure 9.1 and table 9.1. We mix the third index into the symmetric 2-index subspace using the invariant matrix

$$\mathbf{Q} = \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12} = \quad . \tag{9.7}$$

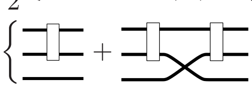


Here projection operators $\mathbf{S}_{12}$ ensure the restriction to the 2-index symmetric subspace, and the transposition $\sigma_{(23)}$ mixes in the third index. To find the characteristic equation for $\mathbf{Q}$, we compute $\mathbf{Q}^2$:

$$\mathbf{Q}^2 = \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12} = \frac{1}{2}\left\{\mathbf{S}_{12} + \mathbf{S}_{12}\sigma_{(23)}\mathbf{S}_{12}\right\} = \frac{1}{2}\mathbf{S}_{12} + \frac{1}{2}\mathbf{Q}$$



$$= \quad = \frac{1}{2}\left\{ \quad + \quad \right\}.$$

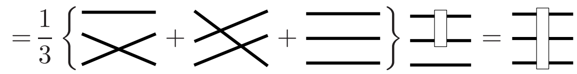


Hence, $\mathbf{Q}$ satisfies

$$(\mathbf{Q}-1)(\mathbf{Q}+1/2)\mathbf{S}_{12}=0\,, \tag{9.8}$$

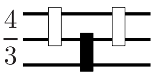
and the corresponding projection operators (3.48) are

$$\mathbf{P}_1 = \frac{\mathbf{Q}+\frac{1}{2}\mathbf{1}}{1+\frac{1}{2}}\mathbf{S}_{12} = \frac{1}{3}\left\{\sigma_{(23)}+\sigma_{(123)}+\mathbf{1}\right\}\mathbf{S}_{12}=\mathbf{S}$$



(9.9)

$$\mathbf{P}_2 = \frac{\mathbf{Q}-\mathbf{1}}{-\frac{1}{2}-1}\mathbf{S}_{12} = \frac{4}{3}\mathbf{S}_{12}A_{23}\mathbf{S}_{12} = \frac{4}{3}$$

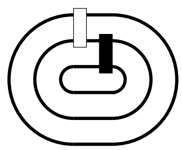


(9.10)

Hence, the symmetric 2-index subspace combines with the third index into a symmetric 3-index subspace (6.13) and a mixed symmetry subspace with dimensions

$$d_1 = \mathrm{tr}\,\mathbf{P}_1 = n(n+1)(n+2)/3! \tag{9.11}$$

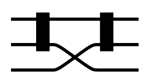
$$d_2 = \mathrm{tr}\,\mathbf{P}_2 = \frac{4}{3}$$



$$= n(n^2-1)/3\,. \tag{9.12}$$

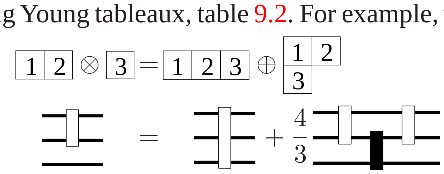
The antisymmetric 2-index subspace can be treated in the same way using the invariant matrix

$$\mathbf{Q} = \mathbf{A}_{12}\sigma_{(23)}\mathbf{A}_{12} =$$



(9.13)

The resulting projection operators for the antisymmetric and mixed symmetry 3-index tensors are given in figure 9.1. Symmetries of the subspace are indicated by the corresponding Young tableaux, table 9.2. For example, we have just constructed



$$\frac{n^2(n+1)}{2} = \frac{n(n+1)(n+2)}{3!} + \frac{n(n^2-1)}{3}\,. \tag{9.14}$$

The projection operators for tensors with up to 4 indices are shown in figure 9.1, and in figure 9.2 the corresponding stepwise reduction of the irreps is given in terms of Young standard tableaux (defined in section 9.3.1).

9.3 YOUNG TABLEAUX

We have seen in the examples of sections. 9.1–9.2 that the projection operators for 2-index and 3-index tensors can be constructed using characteristic equations. For tensors with more than three indices this method is cumbersome, and it is much simpler to construct the projection operators directly from the Young tableaux. In this section we review the Young tableaux and some aspects of symmetric group representations that will be important for our construction of the projection operators in section 9.4.

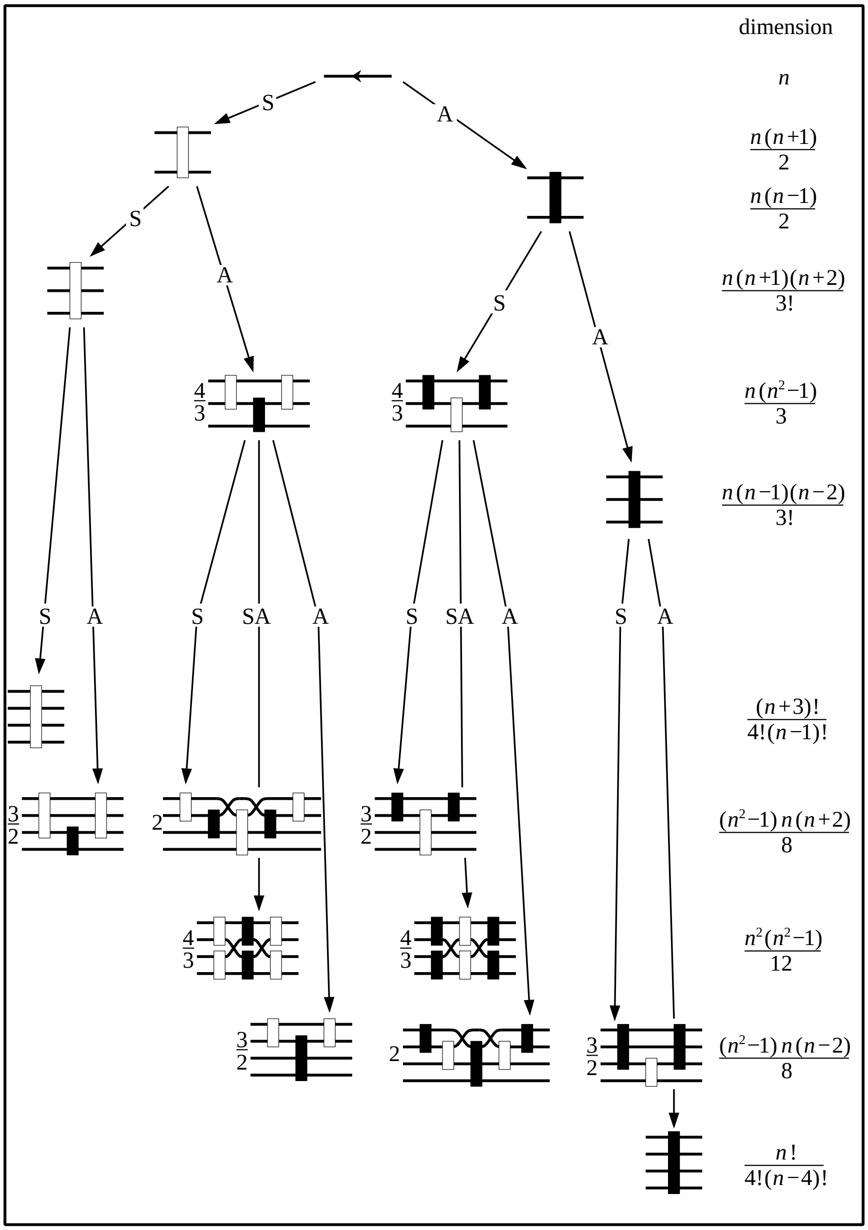


Figure 9.1 Projection operators for 2-, 3-, and 4-index tensors in $U(n)$, $SU(n)$, $n \geq p =$ number of indices.

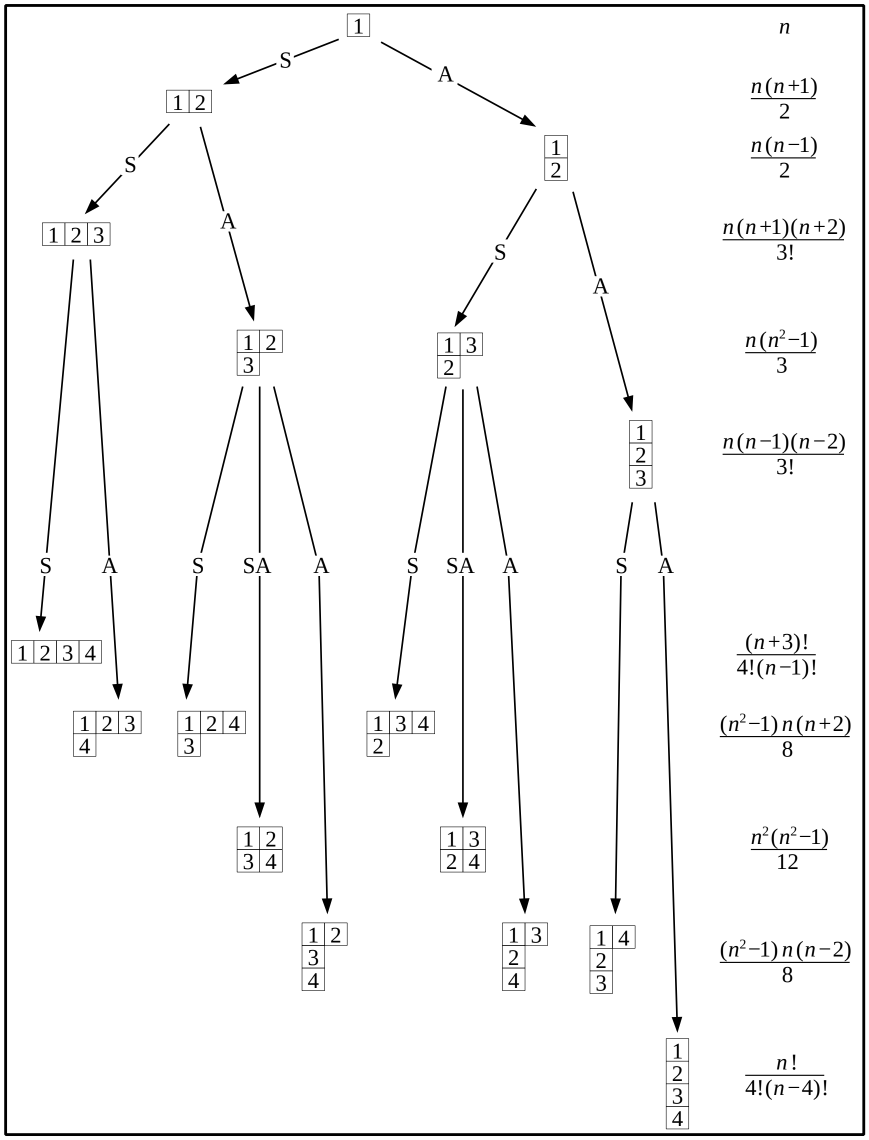
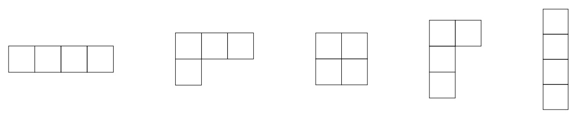


Figure 9.2 Young tableaux for the irreps of the symmetric group for 2-, 3-, and 4-index tensors. Rows correspond to symmetrizations, columns to antisymmetrizations. The reduction procedure is not unique, as it depends on the order in which the indices are combined; this order is indicated by labels 1, 2, 3 , ..., p in the boxes of Young tableaux.

9.3.1 Definitions

Partition k identical boxes into D subsets, and let λ_m, $m = 1, 2, \ldots, D$, be the number of boxes in the subsets ordered so that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_D \geq 1$. Then the partition $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_D]$ fulfills $\sum_{m=1}^{D} \lambda_m = k$. The diagram obtained by drawing the D rows of boxes on top of each other, left aligned, starting with λ_1 at the top, is called a *Young diagram* Y.

Examples:
The ordered partitions for $k = 4$ are $[4]$, $[3, 1]$, $[2, 2]$, $[2, 1, 1]$ and $[1, 1, 1, 1]$. The corresponding Young diagrams are


.

Inserting a number from the set $\{1, \ldots, n\}$ into every box of a Young diagram Y_λ in such a way that numbers increase when reading a column from top to bottom, and numbers do not decrease when reading a row from left to right, yields a *Young tableau* Y_a. The subscript a labels different tableaux derived from a given Young diagram, *i.e.*, different admissible ways of inserting the numbers into the boxes.

A *standard tableau* is a k-box Young tableau constructed by inserting the numbers $1, \ldots, k$ according to the above rules, but using each number exactly once. For example, the 4-box Young diagram with partition $\lambda = [2, 1, 1]$ yields three distinct standard tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \;,\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \;,\quad \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \;. \tag{9.15}$$

An alternative labeling of a Young diagram are Dynkin labels, the list of numbers b_m of columns with m boxes: $(b_1 b_2 \ldots)$. Having k boxes we must have $\sum_{m=1}^{k} m b_m = k$. For example, the partition $[4, 2, 1]$ and the labels $(21100\cdots)$ give rise to the same Young diagram, and so do the partition $[2, 2]$ and the labels $(020\cdots)$.

We define the *transpose* diagram Y^t as the Young diagram obtained from Y by interchanging rows and columns. For example, the transpose of $[3, 1]$ is $[2, 1, 1]$,

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}^{\,t} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} \;,$$

or, in terms of Dynkin labels, the transpose of $(210\ldots)$ is $(1010\ldots)$.

The Young tableaux are useful for labeling irreps of various groups. We shall use the following facts (see for instance ref. [153]):

1. The k-box *Young diagrams* label all irreps of the symmetric group S_k.

2. The *standard tableaux* of k-box Young diagrams with no more than n rows label the irreps of $GL(n)$, in particular they label the irreps of $U(n)$.

3. The *standard tableaux* of k-box Young diagrams with no more than $n-1$ rows label the irreps of $SL(n)$, in particular they label the irreps of $SU(n)$.

In this section, we consider the Young tableaux for reps of S_k and $U(n)$, while the case of $SU(n)$ is postponed to section 9.8.

9.3.2 Symmetric group S_k

The irreps of the symmetric group S_k are labeled by the k-box Young diagrams. For a given Young diagram, the basis vectors of the corresponding irrep can be labeled by the standard tableaux of Y; consequently the dimension Δ_{Y} of the irrep is the number of standard tableaux that can be constructed from the Young diagram Y. The example (9.15) shows that the irrep $\lambda = [2,1,1]$ of S_4 is 3-dimensional.

As an alternative to counting standard tableaux, the dimension Δ_{Y} of the irrep of S_k corresponding to the Young diagram Y can be computed easily as

$$\Delta_{\mathrm{Y}} = \frac{k!}{|\mathrm{Y}|}\,, \tag{9.16}$$

where the number $|\mathrm{Y}|$ is computed using a "hook" rule: Enter into each box of the Young diagram the number of boxes below and to the right of the box, including the box itself. Then $|\mathrm{Y}|$ is the product of the numbers in all the boxes. For instance,

$$\mathrm{Y} = \begin{array}{|c|c|c|c|}\hline \; & \; & \; & \; \\ \hline \; & \; & \; \\ \cline{1-3} \; & \; \\ \cline{1-2}\end{array} \quad \longrightarrow \quad |\mathrm{Y}| = \begin{array}{|c|c|c|c|}\hline 6 & 5 & 3 & 1 \\ \hline 4 & 3 & 1 \\ \cline{1-3} 2 & 1 \\ \cline{1-2}\end{array} = 6!\,3\,. \tag{9.17}$$

The hook rule (9.16) was first proven by Frame, de B. Robinson, and Thrall [123]. Various proofs can be found in the literature [296, 170, 133, 142, 21]; see also Sagan [303] and references therein.

We now discuss the regular representation of the symmetric group. The elements $\sigma \in S_k$ of the symmetric group S_k form a basis of a $k!$-dimensional vector space V of elements

$$s = \sum_{\sigma \in S_k} s_\sigma\, \sigma \;\in V\,, \tag{9.18}$$

where s_σ are the components of a vector s in the given basis. If $s \in V$ has components (s_σ) and $\tau \in S_k$, then τs is an element in V with components $(\tau s)_\sigma = s_{\tau^{-1}\sigma}$. This action of the group elements on the vector space V defines an $k!$-dimensional matrix representation of the group S_k, the *regular representation*.

The regular representation is reducible, and each irrep λ appears Δ_λ times in the reduction; Δ_λ is the dimension of the subspace V_λ corresponding to the irrep λ. This gives the well-known relation between the order of the symmetric group $|S_k| = k!$ (the dimension of the regular representation) and the dimensions of the irreps,

$$|S_k| = \sum_{\text{all irreps } \lambda} \Delta_\lambda^2\,.$$

Using (9.16) and the fact that the Young diagrams label the irreps of S_k, we have

$$1 = k! \sum_{(k)} \frac{1}{|Y|^2}\,, \tag{9.19}$$

where the sum is over all Young diagrams with k boxes. We shall use this relation to determine the normalization of Young projection operators in appendix B.3.

The reduction of the regular representation of S_k gives a completeness relation,

$$\mathbf{1} = \sum_{(k)} \mathbf{P}_{\mathrm{Y}} \,,$$

in terms of projection operators

$$\mathbf{P}_{\mathrm{Y}} = \sum_{\mathrm{Y}_a \in \mathrm{Y}} \mathbf{P}_{\mathrm{Y}_a} \,.$$

The sum is over all standard tableaux derived from the Young diagram Y. Each $\mathbf{P}_{\mathrm{Y}_a}$ projects onto a corresponding invariant subspace V_{Y_a}: for each Y there are Δ_{Y} such projection operators (corresponding to the Δ_{Y} possible standard tableaux of the diagram), and each of these project onto one of the Δ_{Y} invariant subspaces V_{Y} of the reduction of the regular representation. It follows that the projection operators are orthogonal and that they constitute a complete set.

9.3.3 Unitary group $U(n)$

The irreps of $U(n)$ are labeled by the k-box Young standard tableaux with no more than n rows. A k-index tensor is represented by a Young diagram with k boxes — one typically thinks of this as a k-particle state. For $U(n)$, a 1-index tensor has n-components, so there are n 1-particle states available, and this corresponds to the n-dimensional fundamental rep labeled by a 1-box Young diagram. There are n^2 2-particle states for $U(n)$, and as we have seen in section 9.1 these split into two irreps: the symmetric and the antisymmetric. Using Young diagrams, we write the reduction of the 2-particle system as

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square \\[-4pt] \square\end{array} . \tag{9.20}$$

Except for the fully symmetric and the fully antisymmetric irreps, the irreps of the k-index tensors of $U(n)$ have mixed symmetry. Boxes in a row correspond to indices that are symmetric under interchanges (symmetric multiparticle states), and boxes in a column correspond to indices antisymmetric under interchanges (antisymmetric multiparticle states). Since there are only n labels for the particles, no more than n particles can be antisymmetrized, and hence only standard tableaux with up to n rows correspond to irreps of $U(n)$.

The number of standard tableaux Δ_{Y} derived from a Young diagram Y is given in (9.16). In terms of irreducible tensors, the Young diagram determines the symmetries of the indices, and the Δ_{Y} distinct standard tableaux correspond to the independent ways of combining the indices under these symmetries. This is illustrated in figure 9.2.

For a given $U(n)$ irrep labeled by some standard tableau of the Young diagram Y, the basis vectors are labeled by the Young tableaux Y_a obtained by inserting the numbers $1, 2, \ldots, n$ into Y in the manner described in section 9.3.1. Thus the dimension of an irrep of $U(n)$ equals the number of such Young tableaux, and we note that all irreps with the same Young diagram have the same dimension. For $U(2)$, the $k = 2$ Young tableaux of the symmetric and antisymmetric irreps are

$$\boxed{1}\boxed{1}\,, \quad \boxed{1}\boxed{2}\,, \quad \boxed{2}\boxed{2}\,, \quad \text{and} \quad \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array}\,,$$

so the symmetric state of $U(2)$ is 3-dimensional and the antisymmetric state is 1-dimensional, in agreement with the formulas (6.4) and (6.15) for the dimensions of the symmetry operators. For $U(3)$, the counting of Young tableaux shows that the symmetric 2-particle irrep is 6-dimensional and the antisymmetric 2-particle irrep is 3-dimensional, again in agreement with (6.4) and (6.15). In section 9.4.3 we state and prove a dimension formula for a general irrep of $U(n)$.

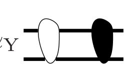
9.4 YOUNG PROJECTION OPERATORS

Given an irrep of $U(n)$ labeled by a k-box standard tableaux Y, we construct the corresponding Young projection operator $\mathbf{P}_\mathrm{Y}$ in birdtrack notation by identifying each box in the diagram with a directed line. The operator $\mathbf{P}_\mathrm{Y}$ is a block of symmetrizers to the left of a block of antisymmetrizers, all imposed on the k lines. The blocks of symmetry operators are dictated by the Young *diagram*, whereas the attachment of lines to these operators is specified by the particular standard tableau.

The Kronecker delta is invariant under unitary transformations: for $U \in U(n)$, we have $(U^\dagger)_a{}^{a'} \delta_{a'}^{b'} U_{b'}{}^b = \delta_a^b$. Consequently, any combination of Kronecker deltas, such as a symmetrizer, is invariant under unitary transformations. The symmetry operators constitute a complete set, so any $U(n)$ invariant tensor built from Kronecker deltas can be expressed in terms of symmetrizers and antisymmetrizers. In particular, the invariance of the Kronecker delta under $U(n)$ transformations implies that the same symmetry group operators that project the irreps of S_k also yield the irreps of $U(n)$.

The simplest examples of Young projection operators are those associated with the Young tableaux consisting of either one row or one column. The corresponding Young projection operators are simply the symmetrizers or the antisymmetrizers respectively. As projection operators for S_k, the symmetrizer projects onto the 1-dimensional subspace corresponding to the fully symmetric representation, and the antisymmetrizer projects onto the fully antisymmetric representation (the alternating representation).

A Young projection operator for a mixed symmetry Young tableau will here be constructed by first antisymmetrizing subsets of indices, and then symmetrizing other subsets of indices; the Young tableau determines which subsets, as will be explained shortly. Schematically,

$$\mathbf{P}_{\mathrm{Y}_a} = \alpha_\mathrm{Y}$$  , (9.21)

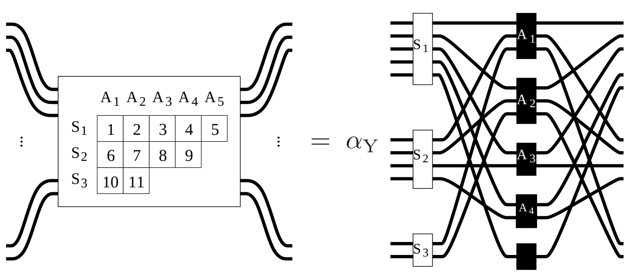
where the white (black) blob symbolizes a set of (anti)symmetrizers. The normalization constant α_Y (defined below) ensures that the operators are idempotent, $\mathbf{P}_{\mathrm{Y}_a}\mathbf{P}_{\mathrm{Y}_b} = \delta_{ab}\mathbf{P}_{\mathrm{Y}_a}$.

This particular form of projection operators is not unique: in section 9.2 we built 3-index tensor Young projection operators that were symmetric under transposition.

The Young projection operators constructed in this section are particularly convenient for explicit $U(n)$ computations, and another virtue is that we can write down the projectors explicitly from the standard tableaux, without having to solve a characteristic equation. For multiparticle irreps, the Young projection operators of this section will generally be different from the ones constructed from characteristic equations (see sections. 9.1–9.2); however, the operators are equivalent, since the difference amounts to a choice of basis.

9.4.1 Construction of projection operators

Let Y_a be a k-box standard tableau. Arrange a set of symmetrizers corresponding to the rows in Y_a, and to the right of this arrange a set of antisymmetrizers corresponding to the columns in Y_a. For a Young diagram Y with s rows and t columns we label the rows S_1, S_2, …, S_s and to the columns A_1, A_2, …, A_t. Each symmetry operator in $\mathbf{P}_\mathrm{Y}$ is associated to a row/column in Y, hence we label a symmetry operator after the corresponding row/column, for example,



(9.22)

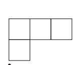
Let the lines numbered 1 to k enter the symmetrizers as described by the numbers in the boxes in the standard tableau and connect the set of symmetrizers to the set of antisymmetrizers in a nonvanishing way, avoiding multiple intermediate lines prohibited by (6.17). Finally, arrange the lines coming out of the antisymmetrizers such that if the lines all passed straight through the symmetry operators, they would exit in the same order as they entered. This ensures that upon expansion of all the symmetry operators, the identity appears exactly once.

We denote by $|\mathrm{S}_i|$ or $|\mathrm{A}_i|$ the *length* of a row or column, respectively, that is the number of boxes it contains. Thus $|\mathrm{A}_i|$ also denotes the number of lines entering the antisymmetrizer A_i. In the above example we have $|\mathrm{S}_1| = 5$, $|\mathrm{A}_2| = 3$, *etc.*

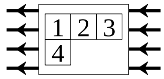
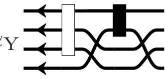
The normalization α_Y is given by

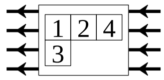
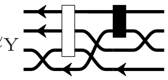
$$\alpha_\mathrm{Y} = \frac{\left(\prod_{i=1}^{s} |\mathrm{S}_i|!\right)\left(\prod_{j=1}^{t} |\mathrm{A}_j|!\right)}{|\mathrm{Y}|}\,, \tag{9.23}$$

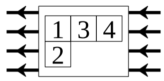
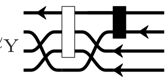
where $|\mathrm{Y}|$ is related through (9.16) to Δ_Y, the dimension of irrep Y of S_k, and is a hook rule S_k combinatoric number. The normalization depends only on the shape of the Young diagram, not the particular tableau.

Example: The Young diagram  tells us to use one symmetrizer of length three, one of length one, one antisymmetrizer of length two, and two of length one.

There are three distinct k-standard arrangements, each corresponding to a projection operator

 $= \alpha_{\mathrm{Y}}$  (9.24)

 $= \alpha_{\mathrm{Y}}$  (9.25)

 $= \alpha_{\mathrm{Y}}$  , (9.26)

where the normalization constant is $\alpha_{\mathrm{Y}} = 3/2$ by (9.23). More examples of Young projection operators are given in section 9.5.

9.4.2 Properties

We prove in appendix B that the above construction yields well defined projection operators. In particular, the internal connection between the symmetrizers and antisymmetrizers is unique up to an overall sign (proof in appendix B.1). We fix the overall sign by requiring that when all symmetry operators are expanded, the identity appears with a positive coefficient. Note that by construction (the lines exit in the same order as they enter) the identity appears exactly once in the full expansion of any of the Young projection operators.

We list here the most important properties of the Young projection operators:

1. The Young projection operators are *orthogonal*: If Y and Z are two distinct standard tableaux, then $\mathbf{P}_{\mathrm{Y}}\mathbf{P}_{\mathrm{Z}} = 0 = \mathbf{P}_{\mathrm{Z}}\mathbf{P}_{\mathrm{Y}}$.

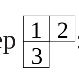
2. With the normalization (9.23), the Young projection operators are indeed *projection operators*, *i.e.*, they are idempotent: $\mathbf{P}_{\mathrm{Y}}^2 = \mathbf{P}_{\mathrm{Y}}$.

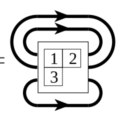
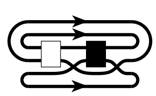
3. For a given k the Young projection operators constitute a complete set such that $\mathbf{1} = \sum \mathbf{P}_{\mathrm{Y}}$, where the sum is over all standard tableaux Y with k boxes.

The proofs of these properties are given in appendix B.

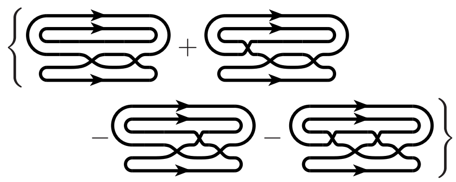
9.4.3 Dimensions of $U(n)$ irreps

The dimension d_{Y} of a $U(n)$ irrep Y can be computed diagrammatically as the trace of the corresponding Young projection operator, $d_{\mathrm{Y}} = \mathrm{tr}\, \mathbf{P}_{\mathrm{Y}}$. Expanding the symmetry operators yields a weighted sum of closed-loop diagrams. Each loop is worth n, and since the identity appears precisely once in the expansion, the dimension d_{Y} of a irrep with a k-box Young tableau Y is a degree k polynomial in n.

Example: We compute we dimension of the $U(n)$ irrep :

$d_{\mathrm{Y}} =$  $= \frac{4}{3}$ 

$$= \frac{4}{3}\left(\frac{1}{2!}\right)^2$$



$$= \frac{1}{3}(n^3 + n^2 - n^2 - n) = \frac{n(n^2-1)}{3} \,. \tag{9.27}$$

In practice, this is unnecessarily laborious. The dimension of a $U(n)$ irrep Y is given by

$$d_Y = \frac{f_Y(n)}{|Y|} \,. \tag{9.28}$$

Here $f_Y(n)$ is a polynomial in n obtained from the Young diagram Y by multiplying the numbers written in the boxes of Y, according to the following rules:

1. The upper left box contains an n.
2. The numbers in a row increase by one when reading from left to right.
3. The numbers in a column decrease by one when reading from top to bottom.

Hence, if k is the number of boxes in Y, $f_Y(n)$ is a polynomial in n of degree k. The dimension formula (9.28) is well known (see for instance ref. [138]).

Example: In the above example with the irrep $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1}\end{array}$, we have

$$d_Y = \frac{f_Y(n)}{|Y|} = \frac{n(n^2-1)}{3}$$

in agreement with the diagrammatic trace calculation (9.27).

Example: With Y = [4,2,1], we have

$$f_Y(n) = \begin{array}{|c|c|c|c|}\hline n & n+1 & n+2 & n+3 \\ \hline n-1 & n & & \\ \cline{1-2} n-2 & & & \\ \cline{1-1}\end{array} = n^2(n^2-1)(n^2-4)(n+3),$$

$$|Y| = \begin{array}{|c|c|c|c|}\hline 6 & 4 & 2 & 1 \\ \hline 3 & 1 & & \\ \cline{1-2} 1 & & & \\ \cline{1-1}\end{array} = 144, \tag{9.29}$$

hence,

$$d_Y = \frac{n^2(n^2-1)(n^2-4)(n+3)}{144} \,. \tag{9.30}$$

Using $d_Y = \mathrm{tr}\,\mathbf{P}_Y$, the dimension formula (9.28) can be proven diagrammatically by induction on the number of boxes in the irrep Y. The proof is given in appendix B.4.

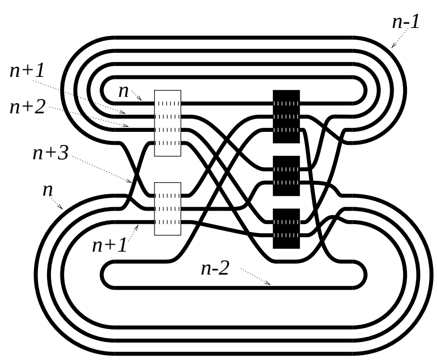
The polynomial $f_{\mathrm{Y}}(n)$ has an intuitive interpretation in terms of strand colorings of the diagram for $\mathrm{tr}\,\mathbf{P}_{\mathrm{Y}}$. Draw the trace of the Young projection operator. Each line is a strand, a closed line, which we draw as passing straight through all of the symmetry operators. For a k-box Young diagram, there are k strands. Given the following set of rules, we count the number of ways to color the k strands using n colors. The top strand (corresponding to the leftmost box in the first row of Y) may be colored in n ways. Color the rest of the strands according to the following rules:

1. If a path, which could be colored in m ways, enters an antisymmetrizer, the lines below it can be colored in $m-1$, $m-2$, … ways.

2. If a path, which could be colored in m ways, enters a symmetrizer, the lines below it can be colored in $m+1$, $m+2$, … ways.

Using this coloring algorithm, the number of ways to color the strands of the diagram is $f_{\mathrm{Y}}(n)$.

Example: For Y = $\begin{array}{|c|c|c|c|}\hline 1&2&3&6\\ \hline 4&5&7\\ \cline{1-3} 8\\ \cline{1-1}\end{array}$, the strand diagram is



(9.31)

Each strand is labeled by the number of admissible colorings. Multiplying these numbers and including the factor $1/|\mathrm{Y}|$, we find

$$d_{\mathrm{Y}} = (n-2)\,(n-1)\,n^2\,(n+1)^2\,(n+2)\,(n+3) \Big/ \begin{array}{|c|c|c|c|}\hline 6&4&3&1\\ \hline 4&2&1\\ \cline{1-3} 1\\ \cline{1-1}\end{array}$$
$$= \frac{n\,(n+1)\,(n+3)!}{2^6\,3^2\,(n-3)!}\,,$$

in agreement with (9.28).

9.5 REDUCTION OF TENSOR PRODUCTS

We now work out several explicit examples of decomposition of direct products of Young diagrams/tableaux in order to motivate the general rules for decomposition

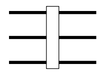
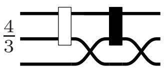
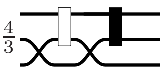
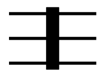
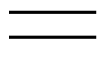
Y_a	$\mathbf{P}_{\mathrm{Y}_a}$	d_{Y_a}
$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\ \hline\end{array}$		$\frac{n(n+1)(n+2)}{6}$
$\begin{array}{\|c\|c\|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}$	$\frac{4}{3}$ 	$\frac{n(n^2-1)}{3}$ (both)
$\begin{array}{\|c\|c\|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}$	$\frac{4}{3}$ 	
$\begin{array}{\|c\|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}$		$\frac{(n-2)(n-1)n}{6}$
$\boxed{1}\otimes\boxed{2}\otimes\boxed{3}$		n^3

Table 9.1 Reduction of 3-index tensor. The last row shows the direct sum of the Young tableaux, the sum of the dimensions of the irreps adding up to n^3, and the sum of the projection operators adding up to the identity as verification of completeness (3.51).

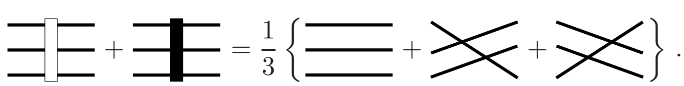
of direct products stated below, in section 9.5.1. We have already treated the decomposition of the 2-index tensor into the symmetric and the antisymmetric tensors, but we shall reconsider the 3-index tensor, since the projection operators are different from those derived from the characteristic equations in section 9.2.

The 3-index tensor reduces to

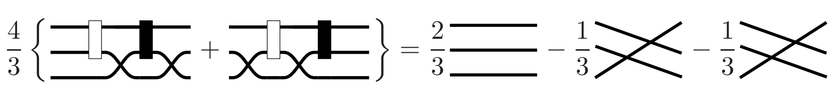
$$\begin{aligned}\boxed{1}\otimes\boxed{2}\otimes\boxed{3} &= \left(\begin{array}{|c|c|}\hline 1&2\\ \hline\end{array}\oplus\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\right)\otimes\boxed{3}\\ &= \begin{array}{|c|c|c|}\hline 1&2&3\\ \hline\end{array}\oplus\begin{array}{|c|c|}\hline 1&2\\ \hline 3\\ \cline{1-1}\end{array}\oplus\begin{array}{|c|c|}\hline 1&3\\ \hline 2\\ \cline{1-1}\end{array}\oplus\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline 3\\ \hline\end{array}\,. \end{aligned}\tag{9.32}$$

The corresponding dimensions and Young projection operators are given in table 9.1. For simplicity, we neglect the arrows on the lines where this leads to no confusion.

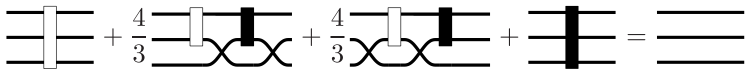
The Young projection operators are orthogonal by inspection. We check completeness by a computation. In the sum of the fully symmetric and the fully antisymmetric tensors, all the odd permutations cancel, and we are left with



Expanding the two tensors of mixed symmetry, we obtain



Adding the two equations we get



,(9.33)

verifying the completeness relation.

For 4-index tensors the decomposition is performed as in the 3-index case, resulting in table 9.2.

Acting with any permutation on the fully symmetric or antisymmetric projection operators gives ± 1 times the projection operator (see (6.8) and (6.18)). For projection operators of mixed symmetry the action of a permutation is not as simple, because the permutations will mix the spaces corresponding to the distinct tableaux. Here we shall need only the action of a permutation within a $3n$-j symbol, and, as we shall show below, in this case the result will again be simple, a factor ± 1 or 0.

9.5.1 Reduction of direct products

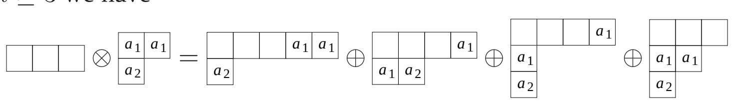
We state the rules for general decompositions of direct products such as (9.20) in terms of Young diagrams:

Draw the two diagrams next to each other and place in each box of the second diagram an a_i, $i = 1, \ldots, k$, such that the boxes in the first row all have a_1 in them, second row boxes have a_2 in them, *etc.* The boxes of the second diagram are now added to the first diagram to create new diagrams according to the following rules:

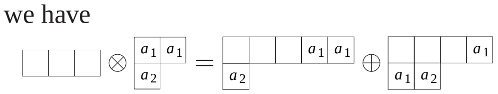
1. Each diagram must be a Young diagram.
2. The number of boxes in the new diagram must be equal to the sum of the number of boxes in the two initial diagrams.
3. For U(n) no diagram has more than n rows.
4. Making a journey through the diagram starting with the top row and entering each row from the right, at any point the number of a_i's encountered in any of the attached boxes must not exceed the number of previously encountered a_{i-1}'s.
5. The numbers must not increase when reading across a row from left to right.
6. The numbers must decrease when reading a column from top to bottom.

Rules 4–6 ensure that states that were previously symmetrized are not antisymmetrized in the product, and vice versa. Also, the rules prevent counting the same state twice.

For example, consider the direct product of the partitions $[3]$ and $[2,1]$. For $U(n)$ with $n \geq 3$ we have



while for $n = 2$ we have



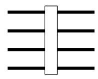
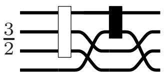
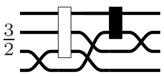
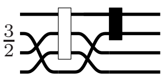
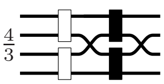
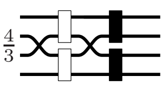
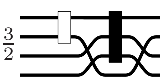
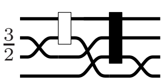
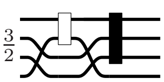
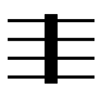
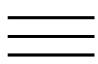
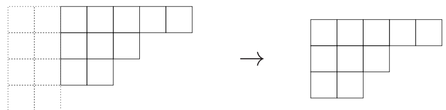
Y_a	$\mathbf{P}_{\mathrm{Y}_a}$	d_{Y_a}
1 2 3 4		$\frac{n(n+1)(n+2)(n+3)}{24}$
1 2 3 / 4	$\frac{3}{2}$ 	
1 2 4 / 3	$\frac{3}{2}$ 	$\frac{(n-1)n(n+1)(n+2)}{8}$
1 3 4 / 2	$\frac{3}{2}$ 	
1 2 / 3 4	$\frac{4}{3}$ 	$\frac{n^2(n^2-1)}{12}$
1 3 / 2 4	$\frac{4}{3}$ 	
1 2 / 3 / 4	$\frac{3}{2}$ 	
1 3 / 2 / 4	$\frac{3}{2}$ 	$\frac{(n-2)(n-1)n(n+1)}{8}$
1 4 / 2 / 3	$\frac{3}{2}$ 	
1 / 2 / 3 / 4		$\frac{n(n-1)(n-2)(n-3)}{24}$
1 ⊗ 2 ⊗ 3 ⊗ 4		n^4

Table 9.2 Reduction of 4-index tensors. Note the symmetry under $n \leftrightarrow -n$.

9.8 $SU(n)$ AND THE ADJOINT REP

The $SU(n)$ group elements satisfy $\det G = 1$, so $SU(n)$ has an additional invariant, the Levi-Civita tensor $\varepsilon_{a_1 a_2 \ldots a_n} = G_{a_1}{}^{a'_1} G_{a_2}{}^{a'_2} \cdots G_{a_n}{}^{a'_n} \varepsilon_{a'_1 a'_2 \ldots a'_n}$. The diagrammatic notation for the Levi-Civita tensors was introduced in (6.27).

While the irreps of $U(n)$ are labeled by the standard tableaux with no more than n rows (see section 9.3), the standard tableaux with a maximum of $n-1$ rows label the irreps of $SU(n)$. The reason is that in $SU(n)$, a column of length n can be removed from any diagram by contraction with the Levi-Civita tensor (6.27). For example, for $SU(4)$

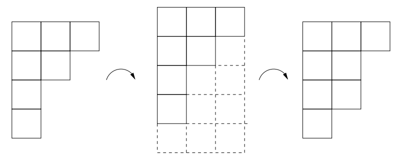


(9.45)

Standard tableaux that differ only by columns of length n correspond to equivalent irreps. Hence, for the standard tableaux labeling irreps of $SU(n)$, the highest column is of height $n-1$, which is also the rank of $SU(n)$. A rep of $SU(n)$, or A_{n-1} in the Cartan classification (table 7.6) is characterized by $n-1$ *Dynkin labels* $b_1 b_2 \ldots b_{n-1}$. The corresponding Young diagram (defined in section 9.3.1) is then given by $(b_1 b_2 \ldots b_{n-1} 00 \ldots)$, or $(b_1 b_2 \ldots b_{n-1})$ for short.

For $SU(n)$ a column with k boxes (antisymmetrization of k covariant indices) can be converted by contraction with the Levi-Civita tensor into a column of $(n-k)$ boxes (corresponding to $(n-k)$ contravariant indices). This operation associates with each diagram a conjugate diagram. Thus the *conjugate* of a $SU(n)$ Young diagram Y is constructed from the missing pieces needed to complete the rectangle of n rows,

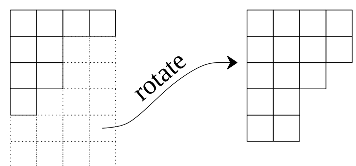
$SU(5)$:



. (9.46)

To find the conjugate diagram, add squares below the diagram of Y such that the resulting figure is a rectangle with height n and width of the top row in Y. Remove the squares corresponding to Y and rotate the rest by 180 degrees. The result is the conjugate diagram of Y. For example, for $SU(6)$ the irrep (20110) has (01102) as its conjugate rep:

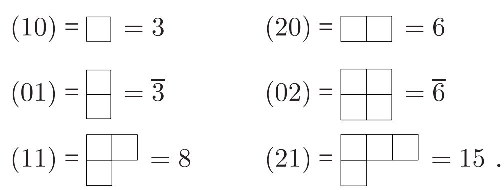
$SU(6)$:



. (9.47)

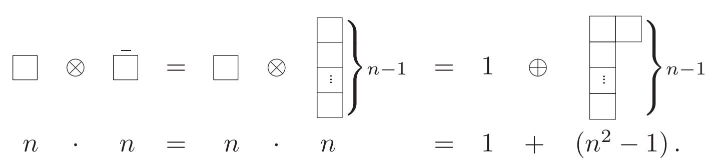
In general, the $SU(n)$ reps $(b_1 b_2 \dots b_{n-1})$ and $(b_{n-1} \dots b_2 b_1)$ are conjugate. For example, $(10 \dots 0)$ stands for the defining rep, and its conjugate is $(00 \dots 01)$, *i.e.*, a column of $n-1$ boxes.

The Levi-Civita tensor converts an antisymmetrized collection of n–1 "in"-indices into 1 "out"-index, or, in other words, it converts an $(n$–$1)$-particle state into a single antiparticle state. We use $\bar{\square}$ to denote the single antiparticle state; it is the conjugate of the fundamental representation $\square$ single particle state. For example, for $SU(3)$ we have



(9.48)

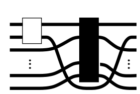
The product of the fundamental rep $\square$ and the conjugate rep $\bar{\square}$ of $SU(n)$ decomposes into a singlet and the *adjoint representation*:



Note that the conjugate of the diagram for the adjoint is again the adjoint.

Using the construction of section 9.4, the birdtrack Young projection operator for the adjoint representation A can be written

$$\mathbf{P}_A = \frac{2(n-1)}{n}$$



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