26.1 Compact groups

All the group theory that we shall need here is given by

> *The Peter-Weyl Theorem, and its corollaries:* A compact Lie group G is completely reducible, its representations are fully reducible (just as in the finite group representation theory), every compact Lie group is a closed subgroup of $U(n)$ for some n, and every continuous, unitary, irreducible representation of a compact Lie group is finite dimensional.

The theory of semisimple Lie groups is elegant, perhaps too elegant. In what follows, we serve group theoretic nuggets in need-to-know portions, offering a pedestrian route through a series of simple examples of familiar aspects of group theory and Fourier analysis, and a high, cyclist road in the text proper.

But main idea is this: the character $\chi^{(m)}(\theta)$ of the Frobenius-Weyl representation theory is a generalization to all compact continuous Lie groups of the weight $e^{i\theta m}$ in the Fourier decomposition of a smooth function on a circle into eigenmodes of translation. mth Fourier component fits m node function around the circle; $(m_1, m_2, \ldots, m_N)$ representation of a compact Lie group fits a corresponding multi-mode function onto the smooth manifold swept out by the action of the group. So a basis for a d-dimensional representation $(m_1, m_2, \ldots, m_N)$ of an N-dimensional compact Lie group is a set of d linearly independent eigenfunctions on the N-dimensional compact group manifold, with m_1, m_2, $\ldots$, m_N 'nodes' along the N directions needed to span the manifold. For a circle this is Fourier analysis; for a sphere these are spherical harmonics, and the Peter-Weyl theorem states that analogous expansion exists for every compact Lie group. We will never need to construct these explicitly.

26.1.1 Group representations

Let q_a be a vector in d-dimensional vector space V, and G be a group of linear transformations

$$q'_a = D(g)_a{}^b q_b \,, \qquad a, b = 1, 2, \ldots, d \,, \quad g \in G$$

(repeated indices summed throughout this chapter). The $[d \times d]$ matrices $D(g)$ form a representation of the group G. Vectors in the dual space $\overline{q}$ transform as

$$q'^a = D(g)^a{}_b \, q^b \,.$$

Tensors transform as

$$h'_{ab}{}^c = D(g)_a{}^f D(g)_b{}^e D(g)^c{}_d \, h_{fe}{}^d \,.$$

A function H is an invariant function if (and only if) for any transformation $g \in G$ and for any set of vectors $\overline{q}, \overline{r}, s, \ldots$

$$H(D(g)^\dagger \overline{q}, D(g)^\dagger \overline{r}, \ldots D(g)s) = H(\overline{q}, \overline{r}, \ldots, s) \,. \tag{26.2}$$

Unitary transformations connected to the identity can be generated by sequences of infinitesimal transformations

$$D(g)_a{}^b \simeq \delta_a^b + i\epsilon_i (T_i)_a^b \qquad \epsilon_i \in \mathbb{R}\,, \quad T_i \text{ hermitian}\,,$$

and $|\epsilon_i| \ll 1$. The N group generators T_a, $a = 1, \cdots, N$ close the Lie algebra of G. (More generally, one also needs to study invariance under discrete coordinate transformations (see chapter 25).

Consider a multilinear invariant function

$$H(\overline{q}, \overline{r}, \ldots, s) = h_{ab\ldots}{}^{\ldots c}\, q^a r^b \ldots s_c$$

In terms of the generators T_i, H is invariant if all generators “annihilate" it, $T_i \cdot h = 0$: [1]

$$(T_i)_a^{a'} h_{a'b\ldots}{}^{c\ldots} + (T_i)_b^{b'} h_{ab'\ldots}{}^{c\ldots} - (T_i)_{c'}^{c} h_{ab\ldots}{}^{c'\ldots} + \ldots \quad = \quad 0\,. \tag{26.3}$$

Vector space V is irreducible if the only invariant subspaces of V under the action of G are (0) and V. If every V on which G acts can be written as a direct sum of irreducible subspaces, then G is completely reducible.

example 26.1
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26.1.2 Group integrals

Consider a group integral of form

$$\int dg\, D(g)_a{}^b D(g)^c{}_d \,, \tag{26.4}$$

where $D(g)_a{}^b$ is a unitary $[d \times d]$ matrix representation of $g \in G$, G a compact Lie group, $D(g)^c{}_d$ is the matrix representation of the action of g on the dual vector space,

$$D(g)^c{}_d = (D(g)^\dagger)_d{}^c \,,$$

and the integration is over the entire range of $g \in G$, G a compact Lie group. For a finite group G with $|G|$ group elements the normalized measure is a discrete sum,

$$d\mu(x) \;=\; \frac{1}{|G|} \sum_g \delta(gx) \,.$$

For continuous groups, the integration measure dg is known as the Haar measure, and, given an explicit parametrization of the group manifold, is explicitly computable (see example 26.4 and example 26.5). However, we do need such explicit parametrizations, as the integral (26.4) over the entire group is defined by two requirements:

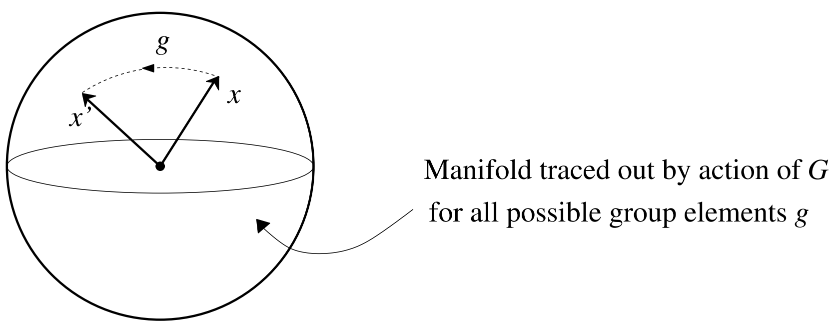
1. *Normalization:* The group average of an scalar quantity is the quantity itself,

$$\int dg = 1 \,. \tag{26.5}$$

2. *Orthonormality of irreducible representations.* How do we define

$$\int dg \, D(g)_a{}^b \; = ?$$

The action of $g \in G$ is to rotate a vector x_a into $x'_a = D(g)_a{}^b x_b$



The averaging smears x in all directions, hence the second integration rule

$$\int dg \, D(g)_a{}^b = 0 \,, \qquad \text{if } D(g) \text{ is } \textit{non-trivial} \text{ representation} \,, \tag{26.6}$$

simply states that the average over all rotations of a vector is zero.

A representation is trivial (a 'singlet') if $D(g) = 1$ for all group elements g. In this case no averaging is taking place, and the first integration rule (26.5) applies.

What happens if we average a bilinear combination of a pair of vectors x, y? There is no reason why such pair should average to zero; for example, we know that the scalar function $|x|^2 = \sum_a x_a x_a^* = x_a x^a$ is invariant under unitary transformations, so it cannot have a vanishing average. Therefore, in general

$$\int dg \, D(g)_a{}^b D(g)^c{}_d \neq 0 \,. \tag{26.7}$$

To get a feeling for what the right-hand side looks like, we recommend that you work out the examples.

example 26.2
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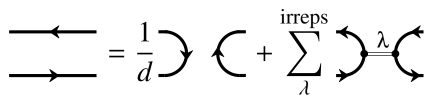
Now let $D(g)$ be any irreducible $[d{\times}d]$ rep. Irreducibility (known in this context as 'Schur's Lemma') means that any invariant $[d{\times}d]$ tensor A^a_b is proportional to δ^a_b. As the only bilinear invariant is δ^a_b, the Clebsch-Gordan series



(26.8)

contains one and only one singlet. Only the singlet survives the group averaging, and

$$\int dg\; D^{(\lambda)}(g)_a{}^d D^{(\lambda)}(g)^b{}_c = \frac{1}{d}\,\delta^d_c \delta^b_a\,. \tag{26.9}$$

is true for any $[d{\times}d]$ *irreducible* rep $D^{(\lambda)}(g)$.

If we take $D^{(\mu)}(g)_\alpha{}^\beta$ and $D^{(\lambda)}(g)_d{}^c$ in inequivalent representations λ, μ (there is no matrix K such that $D^{(\lambda)}(g) = KD^{(\mu)}(g)K^{-1}$ for any $g \in G$), then there is no way of forming a singlet, and

$$\int dg\, D^{(\lambda)}(g)_a{}^d D^{(\mu)}(g)^\beta{}_\alpha = 0 \qquad \text{if} \qquad \lambda \neq \mu\,. \tag{26.10}$$

26.1.3 Characters

The trace of an irreducible $[d{\times}d]$ matrix representation λ of g is called the *character* of the representation:

$$\chi^{(\lambda)}(g) = \mathrm{tr}\, D^{(\lambda)}(g) = D^{(\lambda)}(g)_a{}^a\,. \tag{26.11}$$

The character of the conjugate representation is

$$\chi^{(\lambda)}(g^{-1}) = \mathrm{tr}\, D^{(\lambda)}(g)^\dagger = D^{(\lambda)}(g)^a{}_a = \chi^{(\lambda)}(g)^*\,. \tag{26.12}$$

Contracting (26.8) with two arbitrary invariant $[d{\times}d]$ tensors $h_d{}^a$ and $(f^\dagger)_b{}^c$, we obtain the *character orthonormality relation*

$$\int dg\; \chi^{(\lambda)}(hg)\chi^{(\mu)}(gf) = \delta_{\lambda\mu}\,\frac{1}{d_\lambda}\chi^{(\lambda)}(hf^\dagger) \tag{26.13}$$

The character orthonormality tells us that if two group invariant quantities share a $D^{(\lambda)}(g)D^{(\lambda)}(g^{-1})$ pair, the group averaging sews them into a single group invariant quantity. The replacement of $D^{(\lambda)}(g)_a{}^b$ by the character $\chi^{(\lambda)}(h^{-1}g)$ does not mean that the matrix structure is lost; $D^{(\lambda)}(g)_a{}^b$ can be recovered by differentiating

$$D(g)_a{}^b = \frac{d}{dh_b{}^a}\chi^{(\lambda)}(h^{-1}g)\,. \tag{26.14}$$

The essential group theory we shall need here is most compactly summarized by

The Group Orthogonality Theorem: Let D_μ, $D_{\mu'}$ be two irreducible matrix representations of a compact group G of dimensions d_μ, $d_{\mu'}$,

$$\int dg\; D^{(\mu)}(g)_a{}^b D^{(\mu')}(g^{-1})^{a'}{}_{b'} = \frac{1}{d_\mu}\delta_{\mu,\mu'}\delta_a^{a'}\delta_{b'}^b\,.$$

The new trace formula follows from the full reducibility of representations of a compact group G acting linearly on a vector space V, with irreducible representations labeled by sets of integers $\mu = (\mu_1, \cdots, \mu_N)$, and the vector space V decomposed into invariant subspaces V_μ. For a N-dimensional compact Lie group G the fundamental result is the Weyl full reducibility theorem, with projection operator onto the V_μ irreducible subspace given by [2]

$$P_\mu = d_\mu \int_G g\, \chi^{(\mu)}(g^{-1}) U(g)\,. \tag{26.15}$$

The group elements $g = g(\theta_1, \ldots, \theta_N) = e^{i\theta\cdot T}$ are parameterized by N real numbers $\{\theta_1, \ldots, \theta_N\}$ of finite range, hence designation 'compact'.

Commentary

Remark 26.1. Literature [18] Here we need only basic results, on the level of any standard group theory textbook [14]. This material is covered in any introduction to linear algebra [12, 19, 22] We found Tinkham [29] the most enjoyable as a no-nonsense, the user friendliest introduction to the basic concepts. The construction of projection operators given here is taken from refs. [7–9]. Who wrote this down first we do not know, but we like Harter's exposition [15–17] best. Harter's theory of class algebras offers a more elegant and systematic way of constructing the maximal set of commuting invariant matrices $\mathbf{M}_i$ than the sketch offered here. Bluman and Kumei [2] Chapter 2 offers a clear and pedagogical introduction to Lie groups of transformations. For the Group Orthogonality Theorem see, for example, refs. [9, 30], or consult Googlette.

Remark 26.2. Full reducibility of semisimple Lie groups: The study of integrals over compact Lie groups with respect to Haar measure is important in many areas of mathematics and physics, see Mehta [20]. In 1896-1897 Frobenius introduced notions of 'characters' and group 'representations', and proved the full reducibility of representations of finite groups. The characters $\chi^{(\mu)}(g)$ for all compact semisimple Lie groups were constructed and the full reducibility proven by Weyl [24], extending Cartan's local Lie algebra classification to a global theory of group representations. For the history of this period, see the excellent essay by Hawkins [18].

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26.4 Examples

Example 26.1. Lie algebra. [22] As one does not want the rules to change at every step, the generators T_i are themselves invariant tensors,

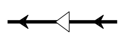
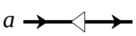
$$(T_i)_{\ b}^{a} = D(g)^{a}{}_{a'} D(g)_{b}{}^{b'} D^{(A)}(g)_{ii'} \, (T_{i'})_{\ b'}^{a'} \,, \qquad (26.27)$$

where $D^{(A)}(g)_{ij}$ is the adjoint $[N\times N]$ matrix representation of $g \in G$. For infinitesimal transformations, $D(g)_a{}^b \simeq \delta_a^b + i\epsilon_i (T_i)_a^b$. The $[d\times d]$ matrices T_i are in general non-commuting, and from (26.3) it follows that they close N-element *Lie algebra*

$$T_i T_j - T_j T_i = iC_{ijk} T_k \qquad i, j, k = 1, 2, ..., N \,,$$

where C_{ijk} are the *structure constants*.

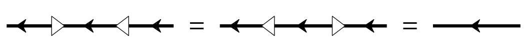
Example 26.2. A group integral for SU(n) $\overline{V} \times V$ space. Let $D(g)$ be the defining $[n\times n]$ matrix representation of SU(n). The defining representation is non-trivial, so it averages to zero by (26.6). The first non-vanishing average involves $D(g)^\dagger$, the matrix representation of the action of g on the conjugate vector space. To avoid dealing with the multitude of dummy indices, we resort to diagrammatic notation: [23]

$D(g)_a{}^\ell = a$  l , $\qquad D(g)^a{}_\ell = a$  l . (26.28)

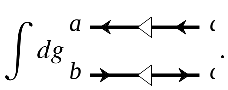
For G the arrows and the triangle point the same way, while for $G^\dagger$ they point the opposite way. Unitarity $D(g)^\dagger D(g) = 1$ is given by

$$D(g)^c{}_a D(g)_c{}^b \;=\; D(g)_a{}^c D(g)^b{}_c \;=\; \delta_a^b \,,$$

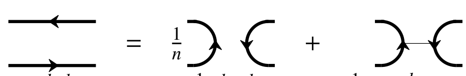
or, diagramatically:

 (26.29)

In this notation, the $D(g)D(g)^\dagger$ integral (26.7) to be evaluated is

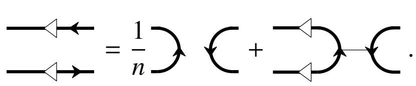
 (26.30)

For SU(n) the $V \otimes \overline{V}$ tensors decompose into the singlet and the adjoint rep

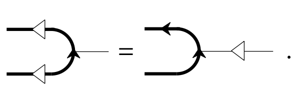


$$\delta_a^d \delta_c^b \;=\; \frac{1}{n}\delta_a^b \delta_c^d \;+\; \frac{1}{a}(T_i)_a^b (T_i)_c^d \,.$$

We multiply (26.30) with the above decomposition of the identity. The unitarity relation (26.29) eliminates G's from the singlet:

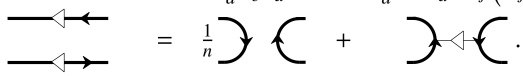
 (26.31)

The generators T_i are invariant tensors, and transform under G according to (26.27). Multiplying by G_{ii}^{-1}, we obtain

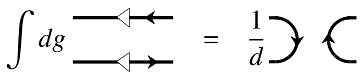
 (26.32)

Hence, the pair $GG^\dagger$ in the defining representation can be traded in for a single G in the adjoint rep

$$D(g)_a{}^d D(g)^b{}_c = \frac{1}{d}\delta^d_c\delta^b_a + \frac{1}{a}(T_i)^b_a G_{ij}\left(T_j\right)^d_c$$



The adjoint representation G_{ij} is non-trivial, so it gets averaged to zero by (26.6). Only the singlet survives



$$\int dg\; D(g)_a{}^d D(g)^b{}_c = \frac{1}{d}\delta^d_c\delta^b_a\,. \qquad (26.33)$$