Instant gratification takes too long.
— Carrie Fisher

25.2.1 Regular representation

Take an arbitrary function $\rho(x)$ defined over the state space $x \in \mathcal{M}$. If the state space is tiled by a fundamental domain $\hat{\mathcal{M}}$ and its copies, function $\rho(x)$ can be written as a $|G|$-dimensional vector of functions, each function defined over the fundamental domain $\hat{x} \in \hat{\mathcal{M}}$ only. The natural choice of a function space basis is the $|G|$-component *regular basis* vector

$$\begin{bmatrix} \rho_1^{reg}(\hat{x}) \\ \rho_2^{reg}(\hat{x}) \\ \vdots \\ \rho_{|G|}^{reg}(\hat{x}) \end{bmatrix} = \begin{bmatrix} \rho(\ D(e)\hat{x}\ ) \\ \rho(D(g_2)\hat{x}) \\ \vdots \\ \rho(D(g_{|G|})\hat{x}) \end{bmatrix}, \tag{25.4}$$

constructed from an arbitrary function $\rho(x)$ defined over the entire state space $\mathcal{M}$, by applying $U(g^{-1})$ to $\rho(\hat{x})$ for each $g \in G$, with state space points restricted to the fundamental domain, $\hat{x} \in \hat{\mathcal{M}}$.

Now apply group action *operator* $U(g)$ to a regular basis vector:

$$U(g)\begin{bmatrix}\rho(\,D(e)\hat{x}\,)\\ \rho(D(g_2)\hat{x})\\ \vdots\\ \rho(D(g_{|G|})\hat{x})\end{bmatrix}=\begin{bmatrix}\rho(\,D(g^{-1})\hat{x}\,)\\ \rho(D(g^{-1}g_2)\hat{x})\\ \vdots\\ \rho(D(g^{-1}g_{|G|})\hat{x})\end{bmatrix}.$$

It acts by permuting the components. (And yes, Mathilde, the pesky g^{-1} is inherited from (25.2), and there is nothing you can do about it.) Thus the action of the *operator* $U(g)$ on a regular basis vector can be represented by the corresponding $[|G|\times|G|]$ permutation *matrix*, called the *left regular representation* $D^{reg}(g)$,

$$U(g)\begin{bmatrix}\rho_1^{reg}(\hat{x})\\ \rho_2^{reg}(\hat{x})\\ \vdots\\ \rho_{|G|}^{reg}(\hat{x})\end{bmatrix}=D^{reg}(g)\begin{bmatrix}\rho_1^{reg}(\hat{x})\\ \rho_2^{reg}(\hat{x})\\ \vdots\\ \rho_{|G|}^{reg}(\hat{x})\end{bmatrix}.$$

A product of two permutations is a permutation, so this is a matrix representation of the group. To compute its entries, write out the matrix multiplication explicitly, labeling the vector components by the corresponding group elements,

$$\rho_b^{reg}(\hat{x})=\sum_a^G D^{reg}(g)_{ba}\,\rho_a^{reg}(\hat{x})\,.$$

A product of two group elements $g^{-1}a$ is a unique element b, so the a_{th} row of $D^{reg}(g)$ is all zeros, except the b_{th} column which satisfies $g=b^{-1}a$. We arrange the columns of the multiplication table by the inverse group elements, as in table 25.1. Setting multiplication table entries with g to 1, and the rest to 0 then defines the regular representation *matrix* $D^{reg}(g)$ for a given g,

$$D^{reg}(g)_{ab}=\delta_{g,b^{-1}a}\,. \tag{25.5}$$

For instance, in the case of the 2-element group $\{e,\sigma\}$ the $D^{reg}(g)$ can be either the identity or the interchange of the two domain labels,

$$D^{reg}(e)=\begin{bmatrix}1&0\\0&1\end{bmatrix}\,,\qquad D^{reg}(\sigma)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\,. \tag{25.6}$$

The multiplication table for D_3 is a more typical, nonabelian group example: see table 25.1. The multiplication tables for C_2 and C_3 are given in table 25.2.

The regular representation of group identity element e is always the identity matrix. As $D^{reg}(g)$ is a permutation matrix, mapping a tile $\hat{\mathcal{M}}_a$ into a different tile $\hat{\mathcal{M}}_{ga}\neq\hat{\mathcal{M}}_a$ if $g\neq e$, only $D^{reg}(e)$ has diagonal elements, and

$$\mathrm{tr}\,D^{reg}(g)=|G|\,\delta_{g,e}\,. \tag{25.7}$$

D_3	e	σ_{12}	σ_{23}	σ_{31}	$C^{1/3}$	$C^{2/3}$
e	e	σ_{12}	σ_{23}	σ_{31}	$C^{1/3}$	$C^{2/3}$
$(\sigma_{12})^{-1}$	σ_{12}	e	$C^{1/3}$	$C^{2/3}$	σ_{23}	σ_{31}
$(\sigma_{23})^{-1}$	σ_{23}	$C^{2/3}$	e	$C^{1/3}$	σ_{31}	σ_{12}
$(\sigma_{31})^{-1}$	σ_{31}	$C^{1/3}$	$C^{2/3}$	e	σ_{12}	σ_{23}
$(C^{1/3})^{-1}$	$C^{2/3}$	σ_{23}	σ_{31}	σ_{12}	e	$C^{1/3}$
$(C^{2/3})^{-1}$	$C^{1/3}$	σ_{31}	σ_{12}	σ_{23}	$C^{2/3}$	e

$$D^{reg}(\sigma_{23}) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} , \quad D^{reg}(C^{1/3}) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Table 25.1: (top) The multiplication table of D_3, the group of symmetries of a triangle. (bottom) By (25.5), the 6 regular representation matrices $D^{reg}(g)$ of dihedral group D_3 have '1' at the location of g in the D_3 multiplication table table 25.1, '0' elsewhere. For example, the regular representation of the action of operators $U(\sigma_{23})$ and $U(C^{2/3})$ on the regular basis (25.4) are shown here.

25.2.2 Irreps: to get invariants, average

A representation $D^{(\mu)}(g)$ acting on d_μ-dimensional vector space $V^{(\mu)}$ is an *irreducible representation* (*irrep*) of group G if its only invariant subspaces are $V^{(\mu)}$ and the null vector $\{0\}$. To develop a feeling for this, one can train on a number of simple examples, and work out in each case explicitly a similarity transformation S that brings $D^{reg}(g)$ to a block diagonal form

$$S^{-1}D^{reg}(g)S = \begin{bmatrix} D^{(1)}(g) & & \\ & D^{(2)}(g) & \\ & & \ddots \end{bmatrix} \tag{25.8}$$

for every group element g, such that the corresponding subspace is invariant under actions $g \in G$, and contains no further nontrivial subspace within it. For the problem at hand we do not need to construct invariant subspaces $\rho^{(\mu)}(x)$ and $D^{(\mu)}(g)$ explicitly. We are interested in the symmetry reduction of the trace formula, and for that we will need only one simple result (lemma, theorem, whatever): the regular representation of a finite group contains all of its irreps μ, and its trace is given by the sum

$$\mathrm{tr}\, D^{reg}(g) = \sum_\mu d_\mu \chi^{(\mu)}(g) \,, \tag{25.9}$$

where d_μ is the dimension of irrep μ, and the characters $\chi^{(\mu)}(g)$ are numbers *intrinsic* to the group G that have to be tabulated only once in the history of humanity. And they all have been. The finiteness of the number of irreps and their dimensions d_μ follows from the dimension sum rule for $\mathrm{tr}\, D^{reg}(e)$, $|G| = \sum d_\mu^2$.

The simplest example is afforded by the 1-dimensional subspace (irrep) given by the fully symmetrized average of components of the regular basis function

$\rho^{reg}(x)$

$$\rho^{(A_1)}(x) = \frac{1}{|G|} \sum_{g}^{G} \rho(D(g)\, x)\,.$$

By construction, $\rho^{(A_1)}$ is invariant under all actions of the group, $U(g)\rho^{(A_1)}(x) = \rho^{(A_1)}(x)$. In other words, for every g this is an eigenvector of the regular representation $D^{reg}(g)$ with eigenvalue 1. Other eigenvalues, eigenvectors follow by working out C_3, C_N (discrete Fourier transform!) and D_3 examples.

The beautiful Frobenius 'character orthogonality' theory of irreps (irreducible representations) of finite groups follows, and is sketched here in appendix A25; it says that all other invariant subspaces are obtained by weighted averages ('projections')

$$\rho^{(\mu)}(x) = \frac{d_\mu}{|G|} \sum_{g} \chi^{(\mu)}(g)\, U(g)\, \rho(x) = \frac{d_\mu}{|G|} \sum_{g} \chi^{(\mu)}(g)\, \rho(D(g^{-1})x) \qquad (25.10)$$

The above $\rho^{(A_1)}(x)$ invariant subspace is a special case, with all $\chi^{(A_1)}(g) = 1$.

By now the group acts in many different ways, so let us recapitulate:

g	abstract group element, multiplies other elements
$D(g)$	$[d \times d]$ state space transformation matrix, multiplies $x \in \mathcal{M}$
$U(g)$	operator, acts on functions $\rho(x)$ defined over state space $\mathcal{M}$
$D^{(\mu)}(g)$	$[d_\mu \times d_\mu]$ irrep, acts on invariant subspace $x \in \mathcal{M}^{(\mu)}$
$D^{reg}(g)$	$[|G| \times |G|]$ regular matrix rep, acts on vectors $x \in \mathcal{M}^{reg}$