## 25.1 Transformation of functions

So far we have recast the problem of long time dynamics into language of lin-ear operators acting of functions, simplest one of which is $\rho(x, t)$, the density of trajectories at time $t$. First we will explain what discrete symmetries do to such functions, and then how they affect their evolution in time.

Let $g$ be an *abstract group element* in $G$. For a discrete group a group element is typically indexed by a discrete label, $g = g_j$. For a continuous group it is typically parametrized by a set of continuous parameters, $g = g(\theta_m)$. As discussed on page 182, linear action of a group element $g \in G$ on a state $x \in \mathcal{M}$ is given by its *matrix representation*, a finite non-singular $[d \times d]$ matrix $D(g)$:

$$x \to x' = D(g)\, x\,. \tag{25.1}$$

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How does the group act on a function $\rho$ of $x$? Denote by $U(g)$ the operator $\rho'(x) = U(g)\,\rho(x)$ that returns the transformed function. One *defines* the transformed function $\rho'$ by requiring that it has the same value at $x' = D(g)x$ as the initial function has at $x$,

$$\rho'(x') = U(g)\,\rho(D(g)x) = \rho(x)\,.$$

Replacing $x \to D(g)^{-1}x$, we find that a group element $g \in G$ acts on a function $\rho(x)$ defined on state space $\mathcal{M}$ by its *operator representation*

$$U(g)\,\rho(x) = \rho(D(g)^{-1}x)\,. \tag{25.2}$$

This is the conventional, Wigner definition of the effect of transformations on functions that should be familiar to master quantum mechanicians. Again: $U(g)$ is an '*operator*', not a matrix - it is an operation whose only meaning is exactly what (25.2) says. And yes, Mathilde, the action on the state space points is $D(g)^{-1}x$, not $D(g)x$.

Consider next the effect of two successive transformations $g_1, g_2$:

$$\begin{aligned} U(g_2)U(g_1)\,\rho(x) &= U(g_2)\,\rho(D(g_1)^{-1}x) = \rho(D(g_2)^{-1}D(g_1)^{-1}x) \\ &= \rho(D(g_1g_2)^{-1}x) = U(g)\rho(x)\,. \end{aligned}$$

Hence if $g_1g_2 = g$, we have $U(g_2)U(g_1) = U(g)$: so operators $U(g)$ form a representation of the group.