11.5 Invariant polynomials

> All invariants are expressible in terms of a finite number among them. We cannot claim its validity for every group G; rather, it will be our chief task to investigate for each particular group whether a finite integrity basis exists or not; the answer, to be sure, will turn out affirmative in the most important cases.
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> —Hermann Weyl, a motivational quote on the "so-called first main theorem of invariant theory"

Physical laws should have the same form in symmetry-equivalent coordinate frames, so they are often formulated in terms of functions (Hamiltonians, Lagrangians, $\cdots$) invariant under a given set of symmetries.

Definition: G-invariant function. A function is said to be G-invariant if

$$f(gx) = f(x)\,, x \in \mathcal{M}\,. \tag{11.6}$$

A G-invariant function is constant along the group orbit of x.

Invariant polynomial functions play a particularly important role in invariant theory. The set of all G-invariant polynomial functions of x which is finitely generated, according to the key result of the representation theory of invariant functions is: [13]

Hilbert-Weyl theorem. For a compact group G there exists a finite G-invariant homogenous polynomial basis $\{u_1, u_2, \ldots, u_m\}$, $m \geq d$, such that any G-invariant polynomial can be written as a multinomial

$$h(x) = p(u_1(x), u_2(x), \ldots, u_m(x))\,, \qquad x \in \mathcal{M}\,. \tag{11.7}$$

These polynomials are linearly independent, but can be functionally dependent through nonlinear relations called *syzygies*.

In practice, explicit construction of G-invariant basis can be a laborious undertaking, and we will not take this path except for a few simple low-dimensional cases, such as 'doubled-polar angle representation' (11.13) and the 5-dimensional example of sect. 13.7. We prefer to apply the symmetry to the system as given, rather than undertake a series of nonlinear coordinate transformations that the theorem suggests. (What 'compact' in the above refers to will become clearer after we have discussed continuous symmetries. For now, it suffices to know that any finite discrete group is compact.)