1.7 Thesis Outline

As stated in the research statement on page 63, an effective study of Wilson line correlators (and its (partial) coincidence limits) over $V^{\otimes m} \otimes (V^*)^{\otimes n}$ presupposes knowledge of the singlet states of $\mathsf{SU}(N)$ over $V^{\otimes m} \otimes (V^*)^{\otimes n}$. However, these singlet states are much less known than a glance at the literature may lead us to believe: In practical examples, one usually considers n-point Wilson line correlators for small n, for which the appropriate singlet states can easily be guessed (see for example [71]). There exists an algorithm for constructing the projection operators onto the irreducible representations of $\mathsf{SU}(N)$ over $V^{\otimes m} \otimes (V^*)^{\otimes n}$ (which includes the singlet representations), but this algorithm is useful only for classification purposes, as it is extremely computationally expensive (this will be exemplified in chapter 5, section 5.1.3). We therefore provide an alternative construction algorithm for the singlet projection operators (and thus the singlet states) in part ***I*** of this thesis.

In the present chapter, the birdtrack formalism was already identified as a useful tool for our purposes. In chapter 2, we give (and prove) several easily implementable simplification rules for birdtrack operators comprised of symmetrizers and antisymmetrizers. These rules fall into two classes: *cancellation* rules, which can be used to shorten the birdtrack expression of a particular operator, and *propagation* rules, which allow one to commute certain sets of symmetrizers and antisymmetrizers.[32] The second set of rules is particularly useful for making the Hermiticity of an operator visually apparent.

In chapter 3, these simplification rules will be put to use when we shift our focus to the irreducible representations of $\mathsf{SU}(N)$ over the space $V^{\otimes m}$. We first review the classical methods of constructing the projection operators onto the desired representations due to Young [84] and improved by Littlewood [85]. However, the projectors constructed in this way lack Hermiticity, which is a crucial problem for our purposes. We then present a more modern approach by Keppeler and Sjödahl (KS) [4], which produces Hermitian Young projection operators. However, the KS operators, beyond the most elementary examples, require a large computational effort to construct. The main result of chapter 3 is an alternative construction principle for compact (and thus easily obtainable) Hermitian Young projection operators, based on the measure of lexical disorder (MOLD) of a Young tableau (Theorem 3.5). The *MOLD operators* are completely equivalent to the KS operators and thus inherit all the desired properties of their KS equivalent. The MOLD construction algorithm relies heavily on the simplification rules of chapter 2.

In chapter 4, we will augment the MOLD operators with what we call *transition operators* to constitute a basis for the algebra of primitive invariants of $\mathsf{SU}(N)$ over $V^{\otimes m}$, see section 4.3.2. These transition operators facilitate a change of basis between projection operators corresponding to equivalent irreducible representations of $\mathsf{SU}(N)$ over $V^{\otimes m}$. The highlight of chapter 4 is an easy-to-implement graphical construction method for transition operators directly from the MOLD operators (Theorem 4.5).

Chapter 5 is the heart of this thesis, as we will see all of the work done in chapters 2 to 4 bear fruit when constructing the singlet projection operators of $\mathsf{SU}(N)$ over a mixed product space $V^{\otimes m} \otimes (V^*)^{\otimes n}$. We first remind the reader about the textbook method used to construct the projection operators corresponding to the irreducible representations of $\mathsf{SU}(N)$ over $V^{\otimes m} \otimes (V^*)^{\otimes n}$ (of which the singlet projectors form a subset) in section 5.1. In doing so, we find that this method is beyond laborious, to the point where it becomes essentially unusable in any practical calculation.

In section 5.2 we will present a **general algorithm to construct the singlet projectors in an easy, computationally efficient way**. We begin by discussing that the singlets over $V^{\otimes k} \otimes (V^*)^{\otimes k}$ (note that there are an equal number of factors V and V^* in this product space) can be found by simply bending the basis elements of the algebra of invariants of $\mathsf{SU}(N)$ over $V^{\otimes k}$ in section 5.2.1. Since the MOLD projection and transition operators span this algebra, they are prime candidates for the construction of singlets (*c.f.* Theorem 5.2).

It is then argued that all the singlet projectors constructed in this way project onto equivalent representations, and we give a general construction principle for the transition operators between singlets (Theorem 5.3). As an example, we construct and examine the singlet projectors and transition operators of $\mathsf{SU}(N)$ over $V^{\otimes 3} \otimes (V^*)^{\otimes 3}$ in various bases, most of which were obtained by bending elements of bases of the algebra of invariants of $\mathsf{SU}(N)$ over $V^{\otimes 3}$.

Section 5.2.2 moves on to the more general product space $V^{\otimes m} \otimes (V^*)^{\otimes n}$. We explain that these projectors are singlets only for certain values of N and that, for this choice of N, they are completely equivalent to the singlets of $\mathsf{SU}(N)$ over $V^{\otimes \alpha} \otimes (V^*)^{\otimes \alpha}$ for a particular integer α (*c.f.* Theorem 5.4).

In light of the fact that we wish to use singlets to study Wilson line correlators, and thus infer properties about the parametrization of the JIMWLK equation (*c.f.* section 1.6.2), we examine the Wilson line correlators over the $3q+3\bar{q}$-algebra in section 5.3.2. In analogy to the discussion of the $2q+2\bar{q}$ correlators in section 1.6.2, we will study various coincidence limits between Wilson lines. In the $3q+3\bar{q}$ example, we will find a nested hierarchy of smaller correlators as limiting cases of larger ones.

In the course of this PhD project, many "incidental" results pertaining to the representation theory of $\mathsf{SU}(N)$ over $V^{\otimes m} \otimes (V^*)^{\otimes n}$ were obtained. These results are given in part ***II*** of this thesis. Most notably, a counting argument for the number of irreducible representations of $\mathsf{SU}(N)$ over $V^{\otimes m} \otimes (V^*)^{\otimes n}$ was found, by establishing a relation between the Hermitian projection operators and the Hermitian primitive invariants of $\mathsf{SU}(N)$ over $V^{\otimes m} \otimes (V^*)^{\otimes n}$ (Theorem 6.2). This theorem gives rise to numerous other results, which will be given in chapter 6. In chapter 7, these results are exemplified: We examine the projection and transition operators (in a particular basis) of $\mathsf{SU}(N)$ over all Fock spaces $V^{\otimes m} \otimes (V^*)^{\otimes n}$ such that $m+n=4$. At the end of the chapter, the special case $N=2$ (relating to the theory of spin) is discussed. Chapter 8 lists a multitude of theorems pertaining to the traces of primitive invariants of $\mathsf{SU}(N)$ over $V^{\otimes (m+n)}$ and $V^{\otimes m} \otimes (V^*)^{\otimes n}$.

We end this thesis with a discussion on possible future research projects. Chapter 9 focuses on the mathematical aspects of this thesis. We list several possible research directions, following on from the results of this thesis, in the pursuit of a full mathematical theory. Chapter 10 discusses possible future research in a multitude of fields in high energy QCD. Such fields include transverse–momentum–dependent parton distributions (TMDs), energy loss, and the parametrization of the JIMWLK equation itself.

Symmetry Implications for Wilson Line Correlators in QCD at High Energies

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