6 Representation theory of the special unitary group $\mathsf{SU}(N)$

6.1 Schur-Weyl duality — an overview

The Schur-Weyl duality is a powerful tool in representation theory that allows one to put the irreducible representations of the general linear group $\mathsf{GL}(N)$ (*c.f.* Definition 6.1) a vector space $V^{\otimes n}$ with $\dim V = N$ into 1-to1 correspondence to the irreducible representations of the group S_n on $V^{\otimes n}$. In particular, it turns out that $V^{\otimes n}$ decomposes as

$$V^{\otimes n} = \bigoplus_{\lambda} V_\lambda \otimes S_\lambda \ , \qquad \lambda \vdash n \ , \tag{6.1}$$

where V_λ are the irreducible submodules of $\mathsf{GL}(N)$ on $V^{\otimes n}$, and S_λ are the so-called *Specht modules*, which describe the irreducible representations of S_n (*c.f.*, e.g., [4] and other standard textbooks). The underlying reason for this is that the actions of $\mathsf{GL}(N)$ and S_n on $V^{\otimes n}$ commute and, even more, the elements of $\mathsf{GL}(N)$ are a *complete set* of actions that commute with those of S_n and vice

In these lectures, we will go through the main points of the Schur-Weyl duality, paying particular attention to the role the Young projection operators play in the representation theory of $\mathsf{GL}(N)$.

- We will begin be defining the general linear group $\mathsf{GL}(N)$ in section 6.2.
- We will then define what we mean by an *invariant* of $\mathsf{GL}(N)$, and show that these invariants are given by the elements of S_n, section 6.3. In fact, S_n *spans* the algebra of invariants of $\mathsf{GL}(N)$. An important ingredient to seeing this the *double commutant theorem*, *c.f.* section 6.3.1.
- Let $\boldsymbol{v} \in V^{\otimes n}$ be arbitrary. We will show that, for every $\Theta \in \mathcal{Y}_n$, the subspace

  $$Y_\Theta \boldsymbol{v} \tag{6.2}$$

  is invariant and irreducible under the action of S_n. Hence, $Y_\Theta \boldsymbol{v}$ is an irreducible $\mathbb{C}[S_n]$-submodule and therefore corresponds to an irreducible representation of S_n on $V^{\otimes n}$, *c.f.* section 6.4.
- Thereafter, we will show that, for every $\Theta \in \mathcal{Y}_n$, the subspace

  $$Y_\Theta V^{\otimes n} \tag{6.3}$$

  is invariant and irreducible under the action of $\mathsf{GL}(N)$ — the main ingredient to showing this is the fact that S_n spans the algebra of invariants of $\mathsf{GL}(N)$ on $V^{\otimes n}$. This shows that the Young projection operators Y_Θ generate the irreducible ideals (and hence the irreducible representations) of $\mathsf{GL}(N)$ on $V^{\otimes n}$, section 6.5.
- Finally, in section 6.6, we will argue that the irreducible representations of $\mathsf{GL}(N)$ on $V^{\otimes n}$ are precicely those of the special unitary group $\mathsf{SU}(N)$ on $V^{\otimes n}$. In other words, the Young projection operators on $V^{\otimes n}$ give rise to all the irreducible ideals of $\mathsf{SU}(N)$ on $V^{\otimes n}$.
- We will end with an example, constructing the irreducible representations of $\mathsf{GL}(N)$ (hence also $\mathsf{SU}(N)$) and S_n on $V^{\otimes n}$ for $n = N = 3$, in section 6.7.

6.2 Basic definitions

■ **Definition 6.1 – General linear group $\mathsf{GL}(V)$ (or $\mathsf{GL}(N)$):**
Let V be a vector space of dimension N (N not necessarily finite). Consider the subset of $End(V)$ of all invertible endomorphisms of V. This set forms a group called the general linear group on V, *and we denote this group by $\mathsf{GL}(V)$ or sometimes only $\mathsf{GL}(N)$ if the vector space V is clear and we want to make the dimension of V explicit.*

■ **Definition 6.2 – Defining/fundamental representation of $\mathsf{GL}(V)$:**
Let $\mathsf{GL}(V)$ be the general linear group acting on a vector space V. We can define a representation γ as

$$\begin{aligned} \gamma: \quad \mathsf{GL}(V) &\rightarrow End(V) \\ \gamma(g) &\mapsto g \end{aligned} \tag{6.6}$$

since there is a well-defined action of $\mathsf{GL}(V)$ on V. The representation γ is referred to as the defining representation *of $\mathsf{GL}(V)$ and has dimension* $\dim V = N$.

6.3 Invariants of $\mathsf{GL}(N)$

Through the defining representation γ of $\mathsf{GL}(V)$ on V, we can define a representation of $\mathsf{GL}(V)$ on $V^{\otimes n}$ via

$$g\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right) := \gamma(g)v_1 \otimes \gamma(g)v_2 \otimes \cdots \otimes \gamma(g)v_n = gv_1 \otimes gv_2 \otimes \cdots \otimes gv_n \tag{6.7a}$$

Let $\mathcal{A}$ be the symmetric group S_n, and let $\mathcal{B}$ be the general linear group $\mathsf{GL}(N)$. Both these groups have a well-defined action on the vector space $V^{\otimes n}$ (with $\dim V = N$), and therefore both S_n and $\mathsf{GL}(N)$ are subgroups of $\mathrm{End}\,(V^{\otimes n})$. Furthermore, we have seen in the beginning of section 6.3 that the actions of S_n and $\mathsf{GL}(N)$ commute on $V^{\otimes n}$, such that

$$\begin{array}{lccc} & \mathcal{A} = S_n \subset (\mathsf{GL}(N))' = \mathcal{B}' & \text{and} & \mathcal{B} = \mathsf{GL}(N) \subset S_n' = \mathcal{A}' \\ \Rightarrow & \mathcal{A} \subset \mathcal{B}' & \text{and} & \mathcal{B} \subset \mathcal{A}' \end{array} . \tag{6.18}$$

It follows directly from Maschke's Theorem 3.1 that $V^{\otimes n}$ is a *completely reducible* S_n-module. That $V^{\otimes n}$ is also a completely reducible $\mathsf{GL}(N)$-module follows from the *Peter-Weyl Theorem* [16] (which we will state without proof):

Note 6.1: Peter-Weyl Theorem

As we will explain in Note 6.2, $\mathsf{GL}(N)$ is a Lie group, which means that, in particular, it is a differentiable manifold. A manifold is said to be *compact* if it is compact as a topological space, that is every open cover has a finite subcover, *c.f.*, e.g., [17]. (*Very* losely speaking, you may think of a cover of a manifold M as another manifold M' enclosing it. The requirement that every subcover is finite can be thought of that every submanifold $N' \subset M'$ that is also a cover for M is finite, implying that M was finite to start off with. As an example, the unit sphere is a compact manifold, but an infinite plane would not be.) It turns out that the representations of such compact Lie groups are completely reducible:

Theorem 6.2 – Reducible carrier spaces of compact groups (Peter-Weyl [16]):
Let φ be a unitary representation of a compact group G on a complex Hilbert space H. Then H splits into an orthogonal direct sum of irreducible finite-dimensional unitary representations of G.

Now, the group $\mathsf{GL}(N)$ is not compact (as a manifold). However, as we will see in the later section 6.6, its subgroup $\mathsf{SU}(N)$ is compact. Furthermore, as we will argue in section section 6.6, the irreducible representations of $\mathsf{GL}(N)$ are precicely those of $\mathsf{SU}(N)$ and vice versa (*c.f.* Theorem 6.5) by means of the so-called *unitarian trick*, it follows that the Peter-Weyl Theorem 6.2 does indeed apply to the group $\mathsf{GL}(N)$ as well.

If all the things said in this note do not quite make sense to you yet, try re-reading this note *after* you have read section 6.6 — this should clear things up for you.

Therefore $V^{\otimes n}$ is completely reducible as an S_n-module, as well as as a $\mathsf{GL}(N)$-module. Therefore, the Double commutant theorem 6.1 asserts that

$$\begin{array}{lccc} & \mathcal{A} = S_n = \mathcal{A}'' & \text{and} & \mathcal{B} = \mathsf{GL}(N) = \mathcal{B}'' \\ \Rightarrow & \mathcal{A} = \mathcal{A}'' & \text{and} & \mathcal{B} = \mathcal{B}'' \end{array} . \tag{6.19}$$

6.6 The unitarian trick: irreducible representations of $\mathsf{SU}(N)$ from $\mathsf{GL}(N)$

Having established the connection between the irreducible representations of S_n and the irreducible representations of $\mathsf{GL}(N)$ on $V^{\otimes n}$,

Definition 6.4 – Special unitary group $\mathsf{SU}(N)$:
Let $\mathsf{GL}(N)$ be the general linear group on a vector space V with dim$(V) = N$. We define $\mathsf{SU}(N)$ to be the subset of matrices in $\mathsf{GL}(N)$ that are unitary (with respect to the canonical scalar product on V) and have determinant 1,

$$\mathsf{SU}(N) = \left\{ U \in \mathsf{GL}(N) \middle| UU^\dagger = \mathbb{1} \text{ and } \det U = 1 \right\} . \tag{6.41}$$

*It can be shown that $\mathsf{SU}(N)$ is in fact a group (*c.f. *) and we call it the* special unitary group on V*.*

(The term *special* refers to the fact that the elements of $\mathsf{SU}(N)$ are unimodular, i.e. have determinant 1, and *unitary* referes to the property that $UU^\dagger = \mathbb{1}$ for every $U \in \mathsf{SU}(N)$.)

The following intermediate result will turn out to be quite useful when establishing the correspondence between the irreducible representations of $\mathsf{GL}(N)$ and the irreducible representations of $\mathsf{SU}(N)$ on $V^{\otimes n}$:

Proposition 6.3 – Square matrix decomposition:
Let $M^{n\times n}$ be the space of all $n \times n$ matrices with entries in $\mathbb{C}$, and let $H^{n\times n}$ and $A^{n\times n}$ be the spaces of all Hermitian, respectively, anti-Hermitian $n \times n$ matrices with entries in $\mathbb{C}$. Then,

$$M^{n\times n} = H^{n\times n} \oplus A^{n\times n} . \tag{6.44}$$

Proof of Proposition 6.3. It is clear that $H^{n\times n} + A^{n\times n} \subset M^{n\times n}$ since a sum of a Hermitian and an anti-Hermtitian $n \times n$ matrix will still yield an $n \times n$ matrix. Conversely, let $m \in M^{n\times n}$. Then, we can write

$$m = \underbrace{\frac{1}{2}(m + m^\dagger)}_{=:m_+} + \underbrace{\frac{1}{2}(m - m^\dagger)}_{=:m_-} , \tag{6.45}$$

where $m^\dagger$ is the Hermitian conjugate of m. Notice that m_+ is Hermitian and m_- is anti-Hermitian. Thus, we also hace that $M^{n\times n} \subset H^{n\times n} + A^{n\times n}$, implying that

$$M^{n\times n} = H^{n\times n} + A^{n\times n} \tag{6.46}$$

It remains to show that $H^{n\times n} \cap A^{n\times n} = \{0\}$ to obtain the desired result: Let $m \in H^{n\times n} \cap A^{n\times n}$. Since $m \in H^{n\times n}$, m is Hermitian. Furthermore, since also $m \in A^{n\times n}$, m is anti-Hermitian. Therefore,

$$m \overset{m\in H^{n\times n}}{=\!=\!=} m^\dagger \overset{m\in A^{n\times n}}{=\!=\!=} -m \qquad \Rightarrow \qquad m = -m \ . \tag{6.47}$$

However, since all entries in m are elements of $\mathbb{C}$, $m = -m$ only holds for m being the zero matrix. Therefore, $H^{n\times n} \cap A^{n\times n} = \{0\}$, yielding

$$M^{n\times n} = H^{n\times n} \oplus A^{n\times n} \ , \tag{6.48}$$

as required. □

Note 6.2: Generators of a Lie group

Without explicitly saying it, both $\mathsf{GL}(N)$ and $\mathsf{SU}(N)$ are *Lie groups*, which is to say that they are differentiable manifolds that also have a group structure defined on them. As differentiable manifolds, they have a tangent space at the identity called the *Lie algebra*. (The Lie algebras of the general linear group $\mathsf{GL}(N)$ and the special unitary group $\mathsf{SU}(N)$ are denoted as $\mathfrak{gl}(N)$ and $\mathfrak{su}(N)$, respectively.) One can define a map, called the *exponential map* from the Lie algebra $\mathfrak{g}$ to the Lie group G as

$$\begin{aligned} \exp: \quad \mathfrak{g} &\to \mathsf{G} \\ X &\mapsto e^{iX} \end{aligned} \tag{6.49}$$

(the i in the exponent is physics convention). It can be shown that, for certain Lie groups (including $\mathsf{GL}(N)$ and $\mathsf{SU}(N)$, but not $\mathsf{SL}(N)$) this exponential map is *surjective*, implying that every element of the Lie group can be written as an exponential of the corresponding element in the Lie alebra! The Lie algebra is a linear space, and in particular it has a basis $\{T_1, T_2, \ldots, T_{\dim(\mathfrak{g})}\}$ such that every $X \in \mathfrak{g}$ can be written as a direct sum of these basis vectors. Thus, each element g of the Lie group G can be written as an exponential of a weighted sum of the basis vectors $T_1, T_2, \ldots, T_{\dim(\mathfrak{g})}$,

$$g = e^{i\sum_j \omega_j T_j} \ , \tag{6.50}$$

where the ω_j are scalars. Hence, the Lie group G is *generated* by the elements in $\{T_1, T_2, \ldots, T_{\dim(\mathfrak{g})}\}$ and we call the $T_1, T_2, \ldots, T_{\dim(\mathfrak{g})}$ the *generators of the Lie group* G.

For this course, we will not discuss any differential geometry, we will merely use the fact that any element G of a Lie group can be written as an exponential $G = e^{i\sum_j \omega_j T_j}$ where the T_j are the generators of the group. Readers, however, are encouraged to explore the topic more on their own, for example by reading through [17].

Now, notice that both $\mathsf{GL}(N)$ and $\mathsf{SU}(N)$ are matrix groups, and hence the exponential map is given by matrix exponentials. In other words, the generators of both $\mathsf{GL}(N)$ and $\mathsf{SU}(N)$ are square matrices.

By definition of $\mathsf{SU}(N)$, each $U \in \mathsf{SU}(N)$ is unitary, $UU^\dagger = \mathbb{1}$. Writing $U = e^{iX}$ for some $X \in \mathfrak{su}(N)$, we thus have that

$$\mathbb{1} \stackrel{!}{=} UU^\dagger = e^{iX}\left(e^{iX}\right)^\dagger = e^{iX}e^{-iX^\dagger} = e^{i(X-X^\dagger)} \qquad \Rightarrow \qquad X - X^\dagger = 0\ . \tag{6.51}$$

Hence, X is Hermitian, and therefore the generators of $\mathsf{SU}(N)$ must all be Hermitian. In fact, eq. (6.51) shows that any Hermitian matrix, when exponentiated, produces an element of $\mathsf{SU}(N)$. Thus, the subset of generators of $\mathsf{GL}(N)$ that is Hermitian are the generators of $\mathsf{SU}(N)$.

As we have seen in Proposition 6.3, every square matrix can be written as a sum of a Hermitian and an anti-Hermitian matrix. Notice that any Hermitian matrix can be made anti-Hermitian by multiplying it with i (and, obviously, any anti-Hermitian matrix with complex entries can be written as i times a Hermitian matrix). Therefore, the generators of $\mathsf{GL}(N)$ are precicely the generators of $\mathsf{SU}(N)$ plus i times the generators of $\mathsf{SU}(N)$

Therefore, if $\varphi : \mathsf{GL}(N) \to \mathrm{End}\,(W)$ is an irreducible representation of $\mathsf{GL}(N)$ on $W \subset V^{\otimes n}$ (with $\dim V = N \geq n$), and we restrict this representation to the subgroup $\mathsf{SU}(N) \subset \mathsf{GL}(N)$, we merely only act φ on half of the generators of $\mathsf{GL}(N)$.[6] This clearly yields a representation of $\mathsf{SU}(N)$

Suppose now that the resetriction of φ on $\mathsf{SU}(N)$, $\varphi\big|_{\mathsf{SU}(N)} : \mathsf{SU}(N) \to \mathrm{End}\,(W)$ is *not* irreducible as a representation of $\mathsf{SU}(N)$. Then one may decompose the carrier space W as a direct sum $W = W_1 \oplus W_2$, where each W_i is irreducible under the action of $\mathsf{SU}(N)$. Hence, any $U \in \mathsf{SU}(N)$ can be written in the form

$$U := \begin{pmatrix} U_{W_1} & 0 \\ 0 & U_{W_2} \end{pmatrix}\ , \tag{6.52}$$

where U_{W_i} is of size $\dim(W_i) \times \dim(W_i)$, leaves the space $W = W_1 \oplus W_2$ invariant. Since this is true for every $U \in \mathsf{SU}(N)$, the generators $T_1, T_2, \ldots T_{\dim(\mathfrak{su}(N))}$ must block-diagonalize the matrices of $\mathsf{SU}(N)$ on the space W. However, since we just reasoned that the T_j form a complete set of generators of $\mathsf{GL}(N)$, it follows that every $g \in \mathsf{GL}(N)$ can be block-diagonalized on W, yielding W to be reducible. This is a contraditiction as $\varphi : \mathsf{GL}(N) \to W$ is known to be irreducible, and hence the restriction $\varphi\big|_{\mathsf{SU}(N)} : \mathsf{SU}(N) \to \mathrm{End}\,(W)$ must be an irreducible representation of $\mathsf{SU}(N)$ as well.

On the other hand, suppose $\tilde{\varphi} : \mathsf{SU}(N) \to \mathrm{End}(\tilde{W})$ is an irreducible representation of $\mathsf{SU}(N)$ on $\tilde{W} \subset V^{\otimes n}$. This representation can be extended to a representation of $\mathsf{GL}(N)$, $\tilde{\varphi}\big|^{\mathsf{GL}(N)} : \mathsf{GL}(N) \to \mathrm{End}(\tilde{W})$, by including the action on i times the generators. Going through a similar chain of arguments as in the previous paragraph, it can be shown that $\tilde{\varphi}\big|^{\mathsf{GL}(N)}$ is an irreducible representation of $\mathsf{GL}(N)$. Therefore, we have:

Theorem 6.5 – Irreducible representations of $\mathsf{SU}(N)$ and $\mathsf{GL}(N)$:
Any irreducible representation of $\mathsf{GL}(N)$ *is an irreducible representation of* $\mathsf{SU}(N)$ *and vice versa.*

Important: Note that Theorem 6.5 is very particular for the groups $\mathsf{GL}(N)$ and $\mathsf{SU}(N)$. If one were to restrict an irreducible representation of $\mathsf{GL}(N)$ to either the *orthogonal group* $\mathsf{O}(N)$ or the *special orthogonal group* $\mathsf{SO}(N)$ (both are subgroups of $\mathsf{GL}(N)$), then the restricted representation is, in general, no longer irreducible. The reason for this is that both $\mathsf{O}(N)$ and $\mathsf{SO}(N)$ have additional invariants to $\mathsf{GL}(N)$ on $V^{\otimes n}$, therefore

Note 6.3: Invariants of $\mathsf{SU}(N)$

The discussion thus far not only shows that the irreducible representations of $\mathsf{SU}(N)$ are precicely those of $\mathsf{GL}(N)$, but further implies that also the linear invariants of $\mathsf{SU}(N)$ are those of $\mathsf{GL}(N)$,

$$\mathsf{API}\left(\mathsf{GL}(N), V^{\otimes n}\right) = \mathsf{API}\left(\mathsf{SU}(N), V^{\otimes n}\right) = \mathbb{C}[S_n] \ . \tag{6.53}$$

Furthermore, the Young projection operators Y_Θ corresponding to the Young tableaux $\Theta \in \mathcal{Y}_n$ generate the irreducible representations of $\mathsf{SU}(N)$ on $V^{\otimes n}$.

6.7 Example: The irreducible representations of S_3 and $\mathsf{SU}(3)$ on $V^{\otimes 3}$

Consider a 3-dimensional vector space V with basis $\{v_1, v_2, v_3\}$. Forming the tensor product space $V^{\otimes 3}$, the basis of V induces a basis on $V^{\otimes 3}$, where each basis vector of $V^{\otimes 3}$ is of the form

$$v_i \otimes v_j \otimes v_k \qquad \text{for } i,j,k \in \{1,2,3\} \ ; \tag{6.54a}$$

clearly, this basis has size $3^3 = 27$. (In general, if $\dim(V) = N$, the tensor product space $V^{\otimes n}$ has dimension N^n.) Introducing the shorthand notation

$$|ijk\rangle := v_i \otimes v_j \otimes v_k \ , \tag{6.54b}$$

the basis vectors of $V^{\otimes 3}$ are given by

$$\begin{array}{lllllll}
|111\rangle\ , & |112\rangle\ , & |121\rangle\ , & |211\rangle\ , & |122\rangle\ , & |221\rangle\ , & |212\rangle\ , \\
|222\rangle\ , & |113\rangle\ , & |131\rangle\ , & |311\rangle\ , & |133\rangle\ , & |331\rangle\ , & |313\rangle\ , \\
|333\rangle\ , & |223\rangle\ , & |232\rangle\ , & |322\rangle\ , & |233\rangle\ , & |332\rangle\ , & |323\rangle\ , \\
|123\rangle\ , & |132\rangle\ , & |213\rangle\ , & |231\rangle\ , & |312\rangle\ , & |321\rangle\ . &
\end{array} \tag{6.55}$$

As usual, the action of a permutation ρ in S_3 on a (basis) vector in $V^{\otimes 3}$ is realized through the permutation of its tensor indices, for example

$$(123)\,|123\rangle \ = \quad \begin{smallmatrix}1\\2\\3\end{smallmatrix} \ = \ \begin{smallmatrix}3\\1\\2\end{smallmatrix} \ = \ |312\rangle \ , \tag{6.56a}$$

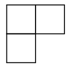
and the action of any $U \in \mathsf{SU}(3)$ on $|ijk\rangle$ yields

$$U\,|ijk\rangle = |U(i)U(j)U(k)\rangle \ . \tag{6.56b}$$

Let us now study the irreducible representations of S_3 and $\mathsf{SU}(3)$ on $V^{\otimes 3}$:

As we have seen in this section, the irreducible representations of S_n and $\mathsf{SU}(N)$ on $V^{\otimes n}$ are generated by the Young projection operators corresponding to the Young tableaux in $\mathcal{Y}_n$. Here, for $n = 3$, we have that

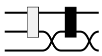
$$\mathcal{Y}_3 = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}\ , \ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\ , \ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\ , \ \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array} \right\} \ . \tag{6.57}$$

First, let us consider the two Young tableaux of shape  ,

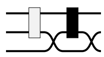
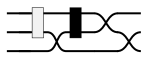
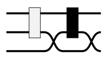
$$\Theta = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \qquad \text{and} \qquad \Phi = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \ . \tag{6.58}$$

According to Theorem 5.2, the Young projection operators Y_Θ and Y_Φ produce an equivalent 2-dimensional irreducible representations of S_3 on $V^{\otimes 3}$. Let us see this explicitly:

First, we need to consider the action of both Y_Θ and Y_Φ on the basis vectors (6.55) of $V^{\otimes 3}$. For the Young projection operator corresponding to the tableau Θ

$Y_\Theta = \frac{4}{3}$  $= \frac{1}{3}\left(\quad - \quad + \quad - \quad \right)$, (6.59)

notice that, for $(13) \in S_3$,

$Y_\Theta(13) = \frac{4}{3}$  $= \frac{4}{3}$  $= -\frac{4}{3}$  $= -Y_\Theta$, (6.60a)

such that, for each vector $|ijk\rangle$, we have that

$$Y_\Theta\,|ijk\rangle = -Y_\Theta(13)\,|ijk\rangle = -Y_\Theta\,|kji\rangle \ , \qquad \text{and hence} \qquad Y_\Theta\,|iji\rangle = 0 \ . \tag{6.60b}$$

Therefore, acting Y_Θ on the basis vectors of $V^{\otimes 3}$ given in eq. (6.55) yields the following 8 *nonzero, linearly independent* vectors:

$$Y_\Theta\,|112\rangle = -Y_\Theta\,|211\rangle = \frac{1}{3}\left(2\,|112\rangle - |211\rangle - |121\rangle\right) \tag{6.61a}$$

$$Y_\Theta\,|113\rangle = -Y_\Theta\,|311\rangle = \frac{1}{3}\left(2\,|113\rangle - |311\rangle - |131\rangle\right) \tag{6.61b}$$

$$Y_\Theta\,|223\rangle = -Y_\Theta\,|322\rangle = \frac{1}{3}\left(2\,|223\rangle - |322\rangle - |232\rangle\right) \tag{6.61c}$$

$$Y_\Theta\,|221\rangle = -Y_\Theta\,|122\rangle = \frac{1}{3}\left(2\,|221\rangle - |122\rangle - |212\rangle\right) \tag{6.61d}$$

$$Y_\Theta\,|331\rangle = -Y_\Theta\,|133\rangle = \frac{1}{3}\left(2\,|331\rangle - |133\rangle - |313\rangle\right) \tag{6.61e}$$

$$Y_\Theta\,|332\rangle = -Y_\Theta\,|233\rangle = \frac{1}{3}\left(2\,|332\rangle - |233\rangle - |323\rangle\right) \tag{6.61f}$$

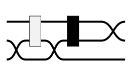
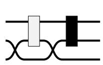
$$Y_\Theta\,|123\rangle = -Y_\Theta\,|321\rangle = \frac{1}{3}\left(|123\rangle - |321\rangle + |213\rangle - |231\rangle\right) \tag{6.61g}$$

$$Y_\Theta\,|132\rangle = -Y_\Theta\,|231\rangle = \frac{1}{3}\left(|132\rangle - |231\rangle + |312\rangle - |321\rangle\right) \ , \tag{6.61h}$$

where

$$Y_\Theta\,|213\rangle = -Y_\Theta\,|312\rangle = Y_\Theta\,|123\rangle - Y_\Theta\,|132\rangle \ . \tag{6.62}$$

Similarly, for the Young projection operator Y_Θ, we notice the following symmetry

$Y_\Phi(12) = \frac{4}{3}$  $= -\frac{4}{3}$  $= -Y_\Phi$, (6.63a)

such that, for each vector $|ijk\rangle$, we have that

$$Y_\Phi\,|ijk\rangle = -Y_\Phi(12)\,|ijk\rangle = -Y_\Phi\,|jik\rangle \ , \qquad \text{and hence} \qquad Y_\Phi\,|iij\rangle = 0 \ . \tag{6.63b}$$

Hence,

$$Y_\Phi\,|121\rangle \;=\; -Y_\Phi\,|211\rangle \;=\; \frac{1}{3}\left(2\,|121\rangle - |211\rangle - |112\rangle\right) \tag{6.64a}$$

$$Y_\Phi\,|131\rangle \;=\; -Y_\Phi\,|311\rangle \;=\; \frac{1}{3}\left(2\,|131\rangle - |311\rangle - |113\rangle\right) \tag{6.64b}$$

$$Y_\Phi\,|232\rangle \;=\; -Y_\Phi\,|322\rangle \;=\; \frac{1}{3}\left(2\,|232\rangle - |322\rangle - |223\rangle\right) \tag{6.64c}$$

$$Y_\Phi\,|212\rangle \;=\; -Y_\Phi\,|122\rangle \;=\; \frac{1}{3}\left(2\,|212\rangle - |122\rangle - |221\rangle\right) \tag{6.64d}$$

$$Y_\Phi\,|313\rangle \;=\; -Y_\Phi\,|133\rangle \;=\; \frac{1}{3}\left(2\,|313\rangle - |133\rangle - |331\rangle\right) \tag{6.64e}$$

$$Y_\Phi\,|323\rangle \;=\; -Y_\Phi\,|233\rangle \;=\; \frac{1}{3}\left(2\,|323\rangle - |233\rangle - |332\rangle\right) \tag{6.64f}$$

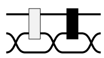
$$Y_\Phi\,|123\rangle \;=\; -Y_\Phi\,|213\rangle \;=\; \frac{1}{3}\left(|123\rangle - |213\rangle + |321\rangle - |312\rangle\right) \tag{6.64g}$$

$$Y_\Phi\,|132\rangle \;=\; -Y_\Phi\,|312\rangle \;=\; \frac{1}{3}\left(|132\rangle - |312\rangle + |231\rangle - |213\rangle\right)\ , \tag{6.64h}$$

and again

$$Y_\Phi\,|231\rangle \;=\; -Y_\Phi\,|321\rangle \;=\; -Y_\Phi\,|123\rangle - Y_\Phi\,|132\rangle\ . \tag{6.65}$$

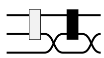
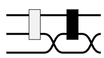
Consider now the irreducible representation of S_3 generated by Y_Θ. If we define and operator $\mathcal{T}_{\Phi\Theta}$ as

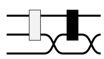
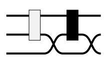
$\mathcal{T}_{\Phi\Theta} := (23)Y_\Theta =$  , (6.66)

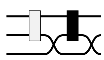
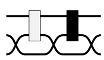
then we have that, for all $\rho \in S_3$,

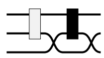
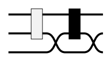
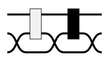
$$\rho Y_\Theta = c_1 Y_\Theta + c_2 \mathcal{T}_{\Phi\Theta}\ , \qquad \text{where } c_1, c_2 \in \mathbb{C}\ , \tag{6.67}$$

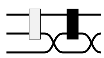
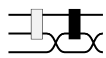
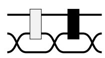
which is to say that Y_Θ generates a 2-dimensional submodule of $V^{\otimes 3}$. Explicitely:

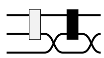
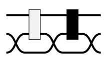
$\mathrm{id}_3 Y_\Theta = \frac{4}{3}\ \cdot$  $= \frac{4}{3}$  $= Y_\Theta$ (6.68a)

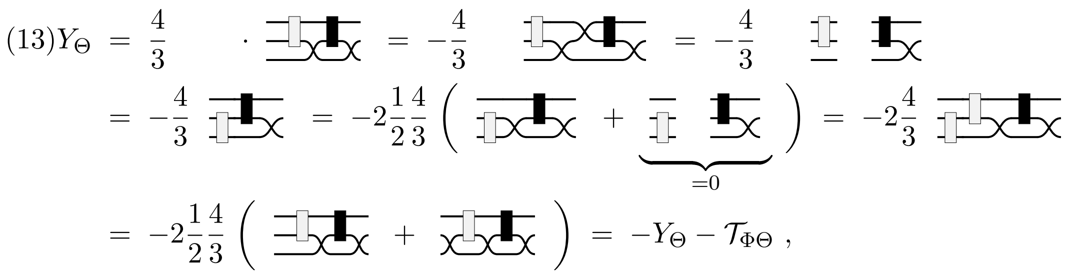
$(12)Y_\Theta = \frac{4}{3}\ \cdot$  $= \frac{4}{3}$  $= Y_\Theta$ (6.68b)

$(23)Y_\Theta = \frac{4}{3}\ \cdot$  $= \frac{4}{3}$  $= \mathcal{T}_{\Phi\Theta}$ (6.68c)

$(13)Y_\Theta = \frac{4}{3}\ \cdot$  $= -\frac{4}{3}$  $- \frac{4}{3}$  $= -Y_\Theta - \mathcal{T}_{\Phi\Theta}$ (6.68d)

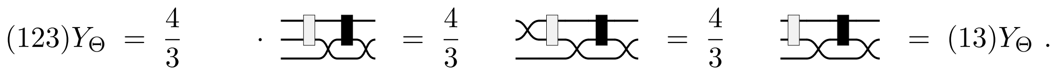
$(123)Y_\Theta = \frac{4}{3}\ \cdot$  $= -\frac{4}{3}$  $- \frac{4}{3}$  $= -Y_\Theta - \mathcal{T}_{\Phi\Theta}$ (6.68e)

$(132)Y_\Theta = \frac{4}{3}\ \cdot$  $= \frac{4}{3}$  $= \mathcal{T}_{\Phi\Theta}$; (6.68f) eq. (6.68d) can be easiest seen as follows:



$$(13)Y_\Theta = \cdots = -2\frac{1}{2}\frac{4}{3}\left(\cdots + \cdots \right) = -Y_\Theta - \mathcal{T}_{\Phi\Theta} \ , \qquad (6.69)$$

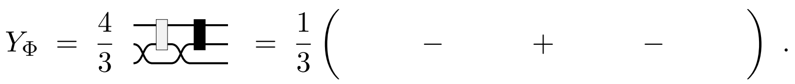
and eq. (6.68e) follows from



$$(123)Y_\Theta = \cdots = (13)Y_\Theta \ . \qquad (6.70)$$

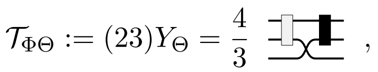
Hence, Y_Θ indeed generates a 2-dimensional S_3-submodule $Y_\Theta \boldsymbol{v}$ of $V^{\otimes 3}$, as claimed in eq. (6.67). Since there exist 8 linearly independent vectors $Y_\Theta \boldsymbol{v}$, this representation is contained 8 times within the regular representation of S_3, and we say that *the 2-dimensional S_3-module generated by Y_Θ has multiplicity* 8.

We will show that the tableau Φ given in eq. (6.58) also gives rise to a 2-dimensional S_3 module with multiplicity 8, which will be shown to be completely equivalent to that generated by Y_Θ: The Young projection operator Y_Φ is given by



$$Y_\Phi = \frac{4}{3} \cdots = \frac{1}{3}\left(\quad - \quad + \quad - \quad \right) . \qquad (6.71)$$

Defining

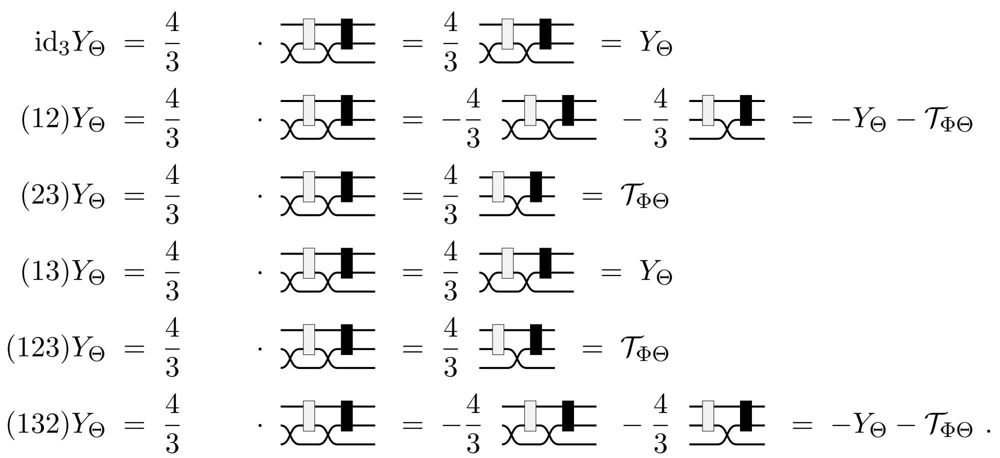


$$\mathcal{T}_{\Phi\Theta} := (23)Y_\Theta = \frac{4}{3} \cdots \ , \qquad (6.72)$$

one may again show that Y_Φ generates a 2-dimensional submodule of $V^{\otimes 3}$ as

$$\rho Y_\Phi = k_1 Y_\Phi + k_2 \mathcal{T}_{\Theta\Phi} \ , \qquad \text{for every } \rho \in S_3, \text{ where } c_1, c_2 \in \mathbb{C} \ . \qquad (6.73)$$

In particular,



$$\mathrm{id}_3 Y_\Theta = \cdots = Y_\Theta \qquad (6.74a)$$

$$(12)Y_\Theta = \cdots = -Y_\Theta - \mathcal{T}_{\Phi\Theta} \qquad (6.74b)$$

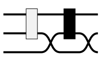
$$(23)Y_\Theta = \cdots = \mathcal{T}_{\Phi\Theta} \qquad (6.74c)$$

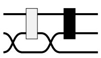
$$(13)Y_\Theta = \cdots = Y_\Theta \qquad (6.74d)$$

$$(123)Y_\Theta = \cdots = \mathcal{T}_{\Phi\Theta} \qquad (6.74e)$$

$$(132)Y_\Theta = \cdots = -Y_\Theta - \mathcal{T}_{\Phi\Theta} \ . \qquad (6.74f)$$

Notice that this agrees with the dimension formula for the irreducible representations of $\mathsf{GL}(N)$ (hence, also $\mathsf{SU}(N)$) given in section 6.5.1, as

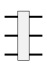
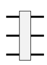
$$\dim Y_\Theta = \mathrm{tr}\left(\frac{4}{3}\right.$$  $$\left.\vphantom{\frac{4}{3}}\right) = \frac{4}{3}\frac{N(N^2-1)}{4} \overset{N=3}{=\!=\!=} 8 \tag{6.80a}$$

$$\dim Y_\Phi = \mathrm{tr}\left(\frac{4}{3}\right.$$  $$\left.\vphantom{\frac{4}{3}}\right) = \frac{4}{3}\frac{N(N^2-1)}{4} \overset{N=3}{=\!=\!=} 8 \ . \tag{6.80b}$$

One can perform a similar analysis for the remaining two tableaux in (6.57). For example, for

$$\Psi = \boxed{1}\boxed{2}\boxed{3} \ , \tag{6.81}$$

the corresponding Young projection operator Y_Ψ is given by

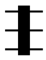
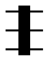
$$Y_\Psi =$$  $$\Rightarrow \qquad \mathrm{tr}\left(\vphantom{\frac{4}{3}}\right.$$  $$\left.\vphantom{\frac{4}{3}}\right) = \frac{N(N+1)(N+2)}{6} \overset{N=3}{=\!=\!=} 10 \ . \tag{6.82}$$

Since Ψ is the unique Young tableau of shape $\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline\end{array}$, Ψ gives rise to 10 equivalent 1-dimensional irreducible representations of S_n on $V^{\otimes 3}$, and a unique 10-dimensional irreducible representation of $\mathsf{SU}(N)$ on $V^{\otimes 3}$.

Lastly, for the tableay

$$\Xi = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}\ , \tag{6.83}$$

the corresponding Young projection operator Y_Ξ is given by

$Y_\Xi =$  $\quad\Rightarrow\quad \mathrm{tr}\Big($  $\Big) = \dfrac{N(N-1)(N-2)}{6} \overset{N=3}{=\!=\!=} 1\ .$ (6.84)

Since Ξ again is the unique Young tableau of shape $\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}$, Ξ gives rise one more 1-dimensional irreducible representations of S_n, which is *inequivalent* to the 1-dimensional representations obtained from Ψ! Furthermore, Y_Ξ a unique 1-dimensional irreducible representation of $\mathsf{SU}(N)$ on $V^{\otimes 3}$ — a 1-dimensional representation of $\mathsf{SU}(N)$ is also called a *singlet representation.*

Notice, however, that for $\dim(V) = N \leq 2$, $\mathrm{tr}\,(Y_\Xi) = 0$, and the tableau Ξ does not give rise to any representations of S_3 or $\mathsf{SU}(N)$ on $V^{\otimes 3}$. In fact, for $N \leq 2$, the Young projection operator Y_Ξ becomes *dimensionally zero*, *c.f.* Note 5.3.

The Special Unitary Group, Birdtracks, and Applications in QCD

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