5 Irreducible representations of S_n and $\mathbb{C}[S_n]$

Let G be a group and let x be a particular element of the group. We define the conjugacy class of x, denoted by x^{G} to be the set

$$\mathsf{x}^{\mathsf{G}} := \left\{\mathsf{g} \in \mathsf{G} \middle| \mathsf{g} = \mathsf{h}\mathsf{x}\mathsf{h}^{-1} \text{ for some } \mathsf{h} \in \mathsf{G}\right\} \tag{5.1}$$

For the symmetric group S_n, it can be shown that every element in a particular conjugacy class have the same cycle structure (*c.f.* Definition 1.2). Conversely, the if two elements of S_n have the same cycle structure, they are in the same conjugacy class

■ **Definition 5.1 – Partition of a natural number:**
Let $n \in \mathbb{N}$, and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be such that

$$\sum_{i=1}^{k} \lambda_i = n \ , \qquad \textit{and} \qquad \lambda_i \geq \lambda_{i+1} \quad \textit{for every } i = 1, 2, \ldots, k-1 \ . \tag{5.8}$$

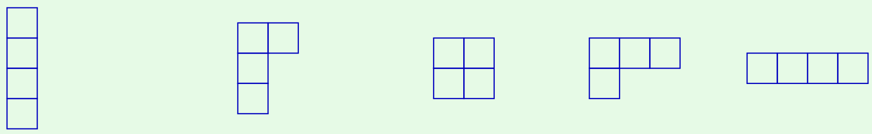
Then, λ is called a partition of n, *and we write $\lambda \vdash n$.*

It is readily seen that the cycle structure of any permutation $\rho \in S_n$ gives a partition of n, and conversely, for any partition λ of n, there exists a cycle in S_n with cycle structure λ. Therefore, the conjugacy classes of S_n correspond uniquely to the partitions of the number n. There is a graphical tool to help keep track of these partitions:

■ **Definition 5.2 – Young diagram:**
Let $n \in \mathbb{N}$ and let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n. The Young diagram $\mathbf{Y}_\lambda$ *corresponding to λ is a planar arrangement of n boxes that are left-aligned and top-aligned, such that the i^{th} row of $\mathbf{Y}_\lambda$ contains exactly λ_i boxes. Furthermore, we say that $\mathbf{Y}_\lambda$ has* size n.

Example 5.1:
The Young diagrams corresponding to the various cycle structures of permutations in S_4 (i.e. partitions of 4) are

$(1,1,1,1)$ $\quad (2,1,1)$ $\quad (2,2)$ $\quad (3,1)$ $\quad (4)$



$4 = 1+1+1+1$ $\quad 4 = 2+1+1$ $\quad 4 = 2+2$ $\quad 4 = 3+1$ $\quad 4 = 4$ $\qquad$ (5.9)

It turns out that the partitions of n have a close connection with the irreducible representations of S_n.

5.1 Equivalent representations & Schur's Lemma

Recall the definition of a representation, in particular an irreducible representation, from section 3.

■ **Definition 5.3 – Equivalent representations:**
Let G be a group and V_1 and V_2 carry two irreducible representations φ_1 and φ_2, respectively, of G,

$$\varphi_1 : \mathsf{G} \to End(V_1) \ , \qquad \textit{and} \qquad \varphi_2 : \mathsf{G} \to End(V_2) \ . \tag{5.10}$$

We say that the representations φ_1 and φ_2 are equivalent, *if there exists an isomorphism $I_{12} : V_2 \to V_1$ such that*

$$I_{12} \circ \varphi_2(\mathsf{g}) \circ I_{12}^{-1} = \varphi_1(\mathsf{g}) \qquad \textit{for every } \mathsf{g} \in \mathsf{G} \ , \tag{5.11}$$

where $\circ$ denotes the composition of linear maps. In the literature, the operator (or map) I_{12} is often also referred to as an intertwining operator.

Now, we are finally in a position to see how the supposed detour via partitions of natural numbers connects to the representation theory of S_n:

Theorem 5.1 – Conjugacy classes give inequivalent irreducible representations:
Let G *be a finite group. Then the conjugacy classes of* G *classify all inequivalent irreducible representations of* G*.*

In particular, if G *is the symmetric group* S_n*, then the Young diagrams of size* n *classify all inequivalent irreducible representations of* S_n*.*

This theorem can easiest be proven using group characters (see, e.g. [11]), which are a powerful tool of group representation theory. However, since in this course we will not be introducing group characters, we leave Theorem 5.1 without proof, but encourage the interested reader to find out more about group characters own his/her own. Alternatively, for the group S_n, one can may also formulate a combinatorial proof as is done in [4].

Note 5.1: Number of inequivalent irreducible representations

Since any finite group G has a finite number of conjugacy classes (this is true since the conjucacy classes partition the group, or can also be seen using *Lagrange's Theorem*), a finite group can only have a finite number of inequivalent irreducible representations!

In particular, the number of inequivalent irreducible representations of S_n is given by $p(n)$, where p is called the partition function, counting the number of partitions of n. However, there is, as of yet, no exact closed form formula for $p(n)$ — finding such a formula is one of the many outstanding problems in number theory.

Example 5.2:

In Example 5.1, we have seen that there are five Young diagrams of size 4. Therefore, we know the group S_4 has five inequivalent irreducible representations, one corresponding to each Young diagram.

Lemma 5.1 – Schur's Lemma:
Let $\mathcal{M}_1$ *and* $\mathcal{M}_2$ *be two irreducible* $\mathbb{F}[\mathsf{G}]$*-modules of a group* G*. Let* $I_{21} : \mathcal{M}_2 \to \mathcal{M}_1$ *be a* G*-homomorphism. Then*

1. I_{12} *is a* G*-isomorphism if and only if* V_1 *and* V_2 *carry equivalent representations of* G*, or*
2. I_{12} *is the zero map.*

Lemma 5.2 – Schur's Lemma (for group representations):
Let $\varphi_1 : \mathsf{G} \to End(V_1)$ and $\varphi_2 : \mathsf{G} \to End(V_2)$ be two irreducible representations of a group G, and let $T : V_2 \to V_1$ be a map satisfying

$$T \circ \varphi_2(\mathsf{g}) = \varphi_1(\mathsf{g}) \circ T \tag{5.16}$$

for every $\mathsf{g} \in \mathsf{G}$. Then

1. *T is invertible or*
2. *T is the zero map.*

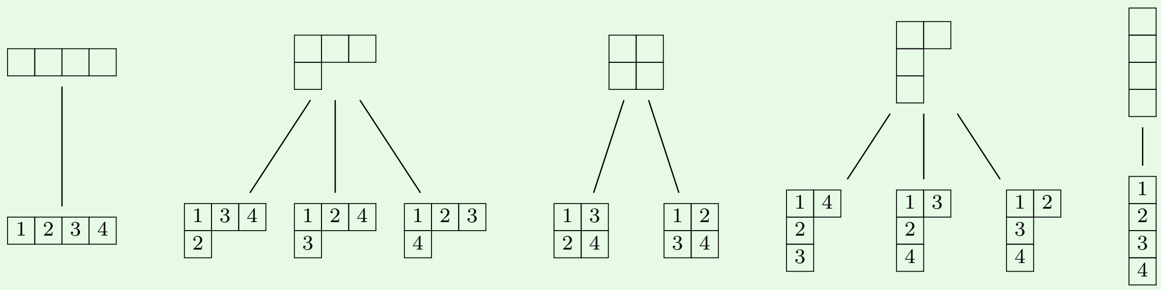
5.2 Young projection operators & irreducible representations of S_n

Young diagrams provide a graphical tool to count the inequivalent irreducible representations of S_n. Granted, Young diagrams are easier to geep track of than partitions of n, but if the story ended here then Young diagrams would only be of little use to us. Luckily for us, this is not the case: Filling the boxes of a Young diagram with numbers in $\mathbb{n} := \{1, 2, \ldots, n\}$ gives us not only a count of *all* irreducible representations of S_n, but, thanks to an algorithm developed by Alfred Young [12], gives immediate access to the primitive idempotents generating the minimal ideals of $\mathbb{C}[S_n]$. Exactly how this happens will be the topic of the present section.

Definition 5.4 – Young tableaux:
Let $\mathbf{Y}$ be a particular Young diagram of size n. A Young tableau *of shape $\mathbf{Y}$ is the diagram $\mathbf{Y}$ where each box is filled with a unique number in $\mathbb{n} = \{1, 2, \ldots, n\}$ such that the numbers increase from left to right and from top to bottom in each row and column.*

We will denote a particular Young tableau with an upper case Greek letter, usually Θ of Φ, and we will denote the Young diagram underlying Θ by $\mathbf{Y}_\Theta$. Furthermore, the set of all Young tableaux of size n (i.e. consisting of n boxes) will be denoted by $\mathcal{Y}_n$.

Example 5.3:
The Young tableaux in $\mathcal{Y}_4$, together with the Young diagram from which they stem, are given by:



In the literature, the presently defined Young tableau is often also referred to as a *standard* Young tableau, where the adjective "standard" refers to the fact that each box is filled with a *unique* integer in $\mathbb{n}$, there may not be any repetitions or numbers missing from $\mathbb{n}$. However, unless we want to emphasize the standardness of the Young tableau, we will simply say Young tableau when we mean a standard Young tableau.

Definition 5.5 – (Anti-)symmetrizers of Young tableaux:
Let $\Theta \in \mathcal{N}$ be a Young tableau with rows $\mathcal{R}_1, \mathcal{R}_2, \ldots \mathcal{R}_s$ and columns $\mathcal{C}_1, \mathcal{C}_2, \ldots \mathcal{C}_t$. Then, we define the product of symmetrizers corresponding to Θ, $\mathbf{S}_\Theta$, to be

$$\mathbf{S}_\Theta := \boldsymbol{S}_{\mathcal{R}_1} \boldsymbol{S}_{\mathcal{R}_2} \cdots \boldsymbol{S}_{\mathcal{R}_s} \ . \tag{5.17a}$$

Similarly, we define the the product of antisymmetrizers corresponding to Θ, $\mathbf{A}_\Theta$, to be

$$\mathbf{A}_\Theta := \boldsymbol{A}_{\mathcal{C}_1} \boldsymbol{S}_{\mathcal{C}_2} \cdots \boldsymbol{S}_{\mathcal{C}_t} \ . \tag{5.17b}$$

Since, by the standardness of Young tableaux, each integer of $\mathbb{n}$ occurs exactly once in Θ, each of the symmetrizers $\boldsymbol{S}_{\mathcal{R}_i}$ in (5.17a) are disjoint, and the same holds true for the antisymmetrizers $\boldsymbol{A}_{\mathcal{C}_j}$ in (5.17b). Therefore, we may also refer to $\mathbf{S}_\Theta$ and $\mathbf{A}_\Theta$ merely as the sets *of symmetrizers, respectively, antisymmetrizers corresponding to Θ.*

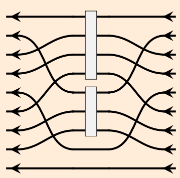
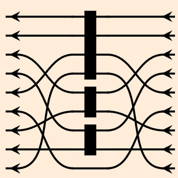
Note 5.2: (Anti-)symmetrizers of Young tableaux in birdtrack notation

Let $\Theta \in \mathcal{Y}_n$ be a particular Young tableau. As was stated in Definition 5.5, the symmetrizers appearing the product $\mathbf{S}_\Theta$ are all disjoint, in that no two symmetrizers in $\mathbf{S}_\Theta$ have common index legs. Therefore, in birdtrack notation, we may draw all of the symmetrizers in $\mathbf{S}_\Theta$ underneath each other, yielding $\mathbf{S}_\Theta$ to be a tower of symmetrizers. The same also may be done with the antisymmetrizers in $\mathbf{A}_\Theta$.
For example, the Young tableau

1	3	4	6
2	7	8	
5			
9			

(5.18a)

has corresponding sets of symmetrizers and antisymmetrizers

$\mathbf{S}_\Theta =$  and $\mathbf{A}_\Theta =$  . (5.18b)

The sets of symmetrizers and antisymmetrizers corresponding to a particular Young tableau $\Theta \in \mathcal{Y}_n$ can be used to create an idempotent operator of $\mathbb{C}[S_n]$. It turns out that the idempotents constructed from Young tableaux, also referred to as *Young projection operators*, give *all* linearly independent idempotents in $\mathbb{C}[S_n]$. Hence, the Young tableaux in $\mathcal{Y}_n$ count, and give direct access to, all irreducible representations of the symmetric group S_n! This is the core message of the following theorem:

Theorem 5.2 – Young projection operators and irreps of S_n:
Let $\Theta, \Phi \in \mathcal{Y}_n$ be two Young tableaux. We define the Young operator *e_Θ to be*

$$e_\Theta := \mathbf{S}_\Theta \mathbf{A}_\Theta \ . \tag{5.19}$$

Then the following statments hold:

1. *The Young operators e_Θ are quasi-idempotent for every $\Theta \in \mathcal{Y}_n$; that is, there exists a nonzero constant $\alpha_\Theta \in \mathbb{C}$ such that*

$$Y_\Theta := \alpha_\Theta e_\Theta = \alpha_\Theta \mathbf{S}_\Theta \mathbf{A}_\Theta \tag{5.20}$$

is idempotent. The operator Y_Θ is referred to as the Young projection operator *corresponding to the tableau Θ.*

2. *The Young projection operators Y_Θ are primitive idempotents, thus generating the minimal ideals of $\mathbb{C}[S_n]$.*

3. *For $\Theta, \Phi \in \mathcal{Y}_n$, the irreducible representations generated by Y_Θ and Y_Φ are equivalent if and only if the tableaux Θ and Φ have the same shape.*

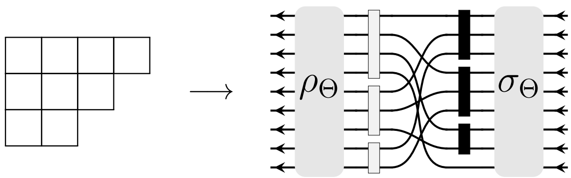
We will delay the proof of Theorem 5.2 to section 5.2.3. For now, let us ponder on what this theorem actually says: As already alluded to previously, Theorem 5.2 states that each Young tableau in $\mathcal{Y}_n$ gives rise to a primitive idempotent $Y_\Theta := \alpha_\Theta \mathbf{S}_\Theta \mathbf{A}_\Theta$ of $\mathbb{C}[S_n]$, where $\alpha_\Theta \in \mathbb{C}$ and $\alpha_\Theta \neq 0$. Thus, the Young tableaux of $\mathcal{Y}_n$ give direct access to the irreducible representations of S_n.

Furthermore, from Theorem 5.1 we know that all inequivalent irreducible representations of S_n are indexed by Young diagrams; part 3 of Theorem 5.2 confirms this by stating that two Young projectors Y_Θ and Y_Φ corresponding to the Young tableaux $\Theta, \Phi \in \mathcal{Y}_n$ generate equivalent representations of S_n if and only if Θ and Φ have the same shape — i.e. if and only if Θ and Φ have the same underlying Young diagram, $\mathbf{Y}_\Theta = \mathbf{Y}_\Phi$.

5.2.1 Structure of Young projection operators & vanishing operators

Let $\Theta \in \mathcal{Y}_n$ be a Young tableau. By definition, each number in $\mathbb{n}$ occurs exactly once in the tableau. Therefore, each symmetrizer in $\mathbf{S}_\Theta$ has at most one leg in common with each antisymmetrizer in $\mathbf{A}_\Theta$.

Let c_1 be the first row in Θ.Tautologically, the elements in c_1 are in the first place in each row, and hence the index lines in the Young projection operator Y_Θ exiting the topmost (first) antisymmetrizer (corresponding to the column c_1) enter the $|c_1|$ symmetrizers in the first place. Similarly, for r_i being the i^{th} row in Θ, the index lines of Y_Θ exiting the i^{th} antisymmetrizer enter the top $|c_i|$ symmetrizers in the i^{th} place.[4] For example,

$\mathbf{Y}_\Theta =$  $\quad$ (5.21)

where the exact form of the permutations ρ_Θ and σ_Θ depend on the *filling* of the Young tableau (i.e. the exact arrangement of numbers in Θ), while the lengths of the symmetrizers and antisymmetrizers, as well as the way in which the index lines connect $\mathbf{A}_\Theta$ to $\mathbf{S}_\Theta$ depends only on the *shape* $\mathbf{Y}_\Theta$ of Θ.

A valid question to ask now is "*Can a Young projection operator ever be zero*"? To answer this question, let us be more precise on what we mean for an operator two be zero. In particular, we distinguish the following cases:

[4] Note that, if this ordering were not already naturally imposed on us, one could always reorder the index lines, as one may factor any permutation out of a symmetrizer at no cost.

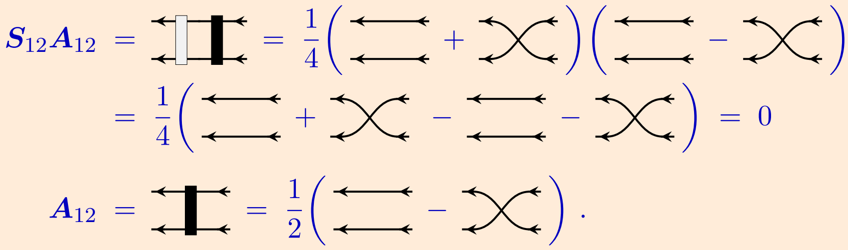
■ **Definition 5.6 – Identically and dimensionally zero operators:**
Let O be an operator acting linearly on aspace $\mathcal{V}$. We say that

1. *O is* identically zero *if $O = 0$, the additive identity in End$(\mathcal{V})$, and*
2. *O is* dimensionally zero *if ker$(O) = \mathcal{V}$.*

Note 5.3: Identically zero and dimensionally zero operators

Note that condition 1 of Definition 5.6 is stronger than condition 2 in that every identically zero operator is dimensionally zero, but there may exist operators whose kernel is the entire space, that are not themselves the additive identity in End$(\mathcal{V})$.

As an example, consider the two operators defined as



(5.22a)

(5.22b)

As we have just seen, the operator $\boldsymbol{S}_{12}\boldsymbol{A}_{12}$ is 0, and hence we say that $\boldsymbol{S}_{12}\boldsymbol{A}_{12}$ is identically zero. On the other hand, $\boldsymbol{A}_{12} \neq 0$, but if we consider the action of $\boldsymbol{A}_{12}$ on $V^{\otimes 2}$ where $\dim(V) = N < 2$, every element of $V^{\otimes 2}$ gets mapped to zero, such that $\ker(\boldsymbol{A}_{12}) = V^{\otimes 2}$. Hence, the operator $\boldsymbol{A}_{12}$ is dimensionally zero but *not* identically zero.

Notice that the nomenclature *dimensionally zero* is inspired by the fact that the space on which the operator O acts is not large enough to support the action: As we have seen in the example (5.22), $\boldsymbol{A}_{12}$ is only dimensionally zero if it acts on $V^{\otimes 2}$ with $\dim(V) = N < 2$. If $\dim(V) = N \geq 2$ then $\boldsymbol{A}_{12}$ is no longer dimensionally zero! In contrast, the operator $\boldsymbol{S}_{12}\boldsymbol{A}_{12} = 0$ on $V^{\otimes 2}$, and hence $\ker(\boldsymbol{S}_{12}\boldsymbol{A}_{12}) = V^{\otimes 2}$, *irrespective* of the dimension of V.

With the considerations in Note 5.3, we can give an alternative definition of identically and dimensionally zero operators in $\mathbb{C}[S_n]$:

■ **Definition 5.7 – Identically and dimensionally zero operators in $\mathbb{C}[S_n]$:**
Let $O \in \mathbb{C}[S_n]$. Then, we can write O as

$$O = \sum_{\sigma \in S_n} \lambda_\sigma \sigma \ , \qquad \lambda_\sigma \in \mathbb{C} \text{ for every } \sigma \in S_n \ . \tag{5.23}$$

We say that

1. *O is* identically zero *if $\lambda_\sigma = 0$ for every $\sigma \in S_n$, and*
2. *O is* dimensionally zero *if ker$(O) = \mathcal{V}$ and there exists at least one $\sigma \in S_n$ such that $\lambda_\sigma \neq 0$.*

5.2.2 Hook length formula

Something that has not been explicitly mentioned in this Theorem is how to find the constant $\alpha_\Theta \in \mathbb{C} \setminus \{0\}$ such that operator $Y_\Theta = \alpha_\Theta e_\Theta$ is idempotent. Luckily however, there exists an easy formula utilizing the *hook rule* to compute α_Θ:

Definition 5.8 – Hook rule & hook length:
Let $\Theta \in \mathcal{Y}_n$ be a particular Young tableau. Its hook length $\mathscr{H}_\Theta$ *is computed using the following* hook rule*:*

Take the Young diagram underlying the tableau Θ, $\mathbf{Y}_\Theta$, and fill each box with the number of boxes lying to the right and underneath it (i.e. the length of the hook whose corner is the cell in question), e.g.

$$\mathbf{Y}_\Theta = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & & \multicolumn{1}{c}{} \\ \cline{1-3} & \multicolumn{3}{c}{} \\ \cline{1-1} & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \qquad \longrightarrow \qquad \begin{array}{|c|c|c|c|}\hline 7 & 4 & 3 & 1 \\ \hline 5 & 2 & 1 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & \multicolumn{3}{c}{} \\ \cline{1-1} 1 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array} \quad . \tag{5.24}$$

The hook length of the tableau Θ is given by the product of all numbers appearing in the resulting tableau; for the example given in eq. (5.24), we have that $\mathscr{H}_\Theta = 7 \cdot 5 \cdot 4 \cdot 3 \cdot 2^2 = 1680$.

The hook length of a Young diagram is defined in an analogous way — one merely foregoes the first step of "deleting the entries" as a Young diagram has no entries in its boxes to begin with. Furthermore, from Definition 5.8, it immeadiately follows that two Young tableaux with the same shape have the same hook lengths.

Theorem 5.3 – Number of Young tableaux of certain shape & normalization constant α_Θ:
Let $\mathbf{Y}$ be a particular Young diagram of size n. Then, the number of Young tableaux with shape $\mathbf{Y}$ is given by

$$\frac{n!}{\mathscr{H}_\mathbf{Y}} \; . \tag{5.25}$$

Let $\Theta \in \mathcal{Y}_n$ be a Young tableau, and denote the length of the i^{th} row by r_i, and the length of the j^{th} column by c_j. Then, the normalization constant α_Θ needed to render $Y_\Theta = \alpha_\Theta e_\Theta$ idempotent is given by

$$\alpha_\Theta = \frac{\prod_i r_i! \cdot \prod_j c_j!}{\mathscr{H}_\mathbf{Y}} \tag{5.26}$$

Theorem 5.3 will be left without proof, but a nice combinatorial proof can be found in [4].

Exercise 5.2: *Write down all Young* diagrams *of size* 6 *(i.e. consisting of six boxes). Compute the Hook length of each diagram. With this information, find the number of Young* tableaux *of size* 6*, i.e. compute $|\mathcal{Y}_6|$.*

Solution: There are, in total, 11 Young diagrams of size 6, namely

$$\ydiagram{6}\ ,\quad \ydiagram{5,1}\ ,\quad \ydiagram{4,1,1}\ ,\quad \ydiagram{4,2}\ ,\quad \ydiagram{3,1,1,1}\ ,$$
$$\ydiagram{3,2,1}\ ,\quad \ydiagram{3,3}\ ,\quad \ydiagram{2,1,1,1,1}\ ,\quad \ydiagram{2,2,1,1}\ ,\quad \ydiagram{2,2,2}\ ,\quad \ydiagram{1,1,1,1,1,1}\ . \tag{5.27}$$

The hook length of each diagram is calculated according to Definition 5.8, for example.

$$\ydiagram{3,2,1} \xrightarrow{\text{hook lengths}} \begin{ytableau} 5 & 3 & 1 \\ 3 & 1 \\ 1 \end{ytableau} \longrightarrow \mathscr{H}_{\ydiagram{3,2,1}} = 5 \cdot 3 \cdot 1 \cdot 3 \cdot 1 \cdot 1 = 45 \tag{5.28a}$$

$$\ydiagram{2,2,2} \xrightarrow{\text{hook lengths}} \begin{ytableau} 4 & 3 \\ 3 & 2 \\ 2 & 1 \end{ytableau} \longrightarrow \mathscr{H}_{\ydiagram{2,2,2}} = 4 \cdot 3 \cdot 3 \cdot 2 \cdot 2 \cdot 1 = 144\ . \tag{5.28b}$$

Continuing in this fashion, we see that the Hook lengths of all diagrams in (5.27) are given by

$$\begin{gathered}
\mathscr{H}_{\ydiagram{6}} = 6! = 720\ , \quad \mathscr{H}_{\ydiagram{5,1}} = 144\ , \quad \mathscr{H}_{\ydiagram{4,1,1}} = 72\ , \\
\mathscr{H}_{\ydiagram{4,2}} = 80\ , \quad \mathscr{H}_{\ydiagram{3,1,1,1}} = 72\ , \quad \mathscr{H}_{\ydiagram{3,2,1}} = 45\ , \quad \mathscr{H}_{\ydiagram{3,3}} = 144\ , \\
\mathscr{H}_{\ydiagram{2,1,1,1,1}} = 144\ , \quad \mathscr{H}_{\ydiagram{2,2,1,1}} = 80\ , \quad \mathscr{H}_{\ydiagram{2,2,2}} = 144\ , \quad \mathscr{H}_{\ydiagram{1,1,1,1,1,1}} = 6! = 720\ .
\end{gathered} \tag{5.29}$$

Theorem 5.3 tells us that the number of Young tableaux corresponding to a particular Young diagram $\mathbf{Y}$ (i.e. tableaux of shape $\mathbf{Y}$) is given by $\frac{n!}{\mathscr{H}_{\mathbf{Y}}}$, where n is the size of the diagram $\mathbf{Y}$. Hence, to find the number of all Young tableaux of size 6, we have to form a sum of the Hook lengths over the Young diagrams of size 6,

$$|\mathcal{Y}_6| = \sum_{\mathbf{Y} \text{ size } 6} \frac{6!}{\mathscr{H}_{\mathbf{Y}}}\ . \tag{5.30a}$$

Hence, we find that

$$\begin{aligned}
|\mathcal{Y}_6| &= \frac{6!}{6!} + \frac{6!}{144} + \frac{6!}{48} + \frac{6!}{80} + \frac{6!}{48} + \frac{6!}{45} + \frac{6!}{144} + \frac{6!}{144} + \frac{6!}{80} + \frac{6!}{144} + \frac{6!}{6!} \\
&= 1 + 5 + 10 + 9 + 10 + 16 + 5 + 5 + 9 + 5 + 1 \\
&= 76\ .
\end{aligned} \tag{5.30b}$$

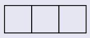
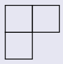
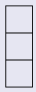
Hence, there are 76 Young tableaux of size 6.

Notice that, if you were only interested in the number of Young tableaux of size n, going this route via the Young diagrams and the hook lengths is not the easiest/quickest way to go, since there is not closed form exact formula for the number of Young diagrams of a certain size (recall Note 5.1).

Luckily however, there exists a closed form formula for the number of Young tableaux, but that is a story for another day....

Exercise 5.3: *Compute all Young projection operators of $\mathbb{C}[S_3]$ (acting on $V^{\otimes 3}$).*

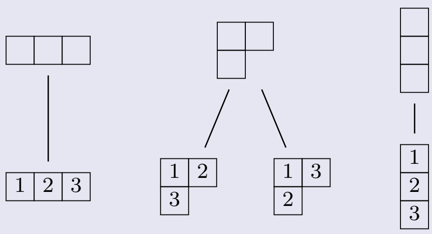
Solution: It is readily seen that the Young diagrams of size 3 are given by

 ,  and  , (5.31a)

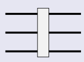
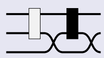
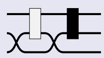
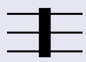
with corresponding hook lengths

$$\mathscr{H}_{\square\square\square} = 3! \,, \qquad \mathscr{H}_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}} = 3 \qquad \text{and} \qquad \mathscr{H}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}} = 3! \,. \tag{5.31b}$$

From Theorem 5.3 we know that the first and last Young diagram in eq. (5.31a) give rise to one Young tableau each, while the middle Young diagram produces two Young tableaux. These tableaux are

 . (5.32)

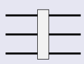
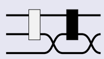
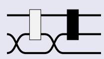
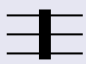
Using the definition of the Young operators $e_\Theta = \mathbf{S}_\Theta \mathbf{A}_\Theta$ given in Theorem 5.2, we find, for every $\Theta \in \mathcal{Y}_3$,

$e_{\boxed{1}\boxed{2}\boxed{3}} =$  , $e_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} =$  , $e_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} =$  , $e_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} =$  . (5.33)

To turn each quasi-idempotent e_Θ into an idempotent $Y_\Theta = \alpha_\Theta e_\Theta$, we compute the normalization constants α_Θ for each $\Theta \in \mathcal{Y}_3$ using Theorem 5.3:

$$\alpha_{\boxed{1}\boxed{2}\boxed{3}} = \frac{3!}{\mathscr{H}_{\square\square\square}} = 1 \,, \qquad \alpha_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \alpha_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = \frac{2!2!}{\mathscr{H}_{\begin{smallmatrix}\square&\square\\\square&\end{smallmatrix}}} = \frac{4}{3} \qquad \text{and} \qquad \alpha_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} = \frac{3!}{\mathscr{H}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}} = 1 \,. \tag{5.34}$$

Therefore, the Young projection operators $Y_\Theta = \alpha_\Theta e_\Theta$ for every $\Theta \in \mathcal{Y}_3$ are given by

$Y_{\boxed{1}\boxed{2}\boxed{3}} =$  , $Y_{\begin{smallmatrix}1&2\\3&\end{smallmatrix}} = \frac{4}{3}$  , $Y_{\begin{smallmatrix}1&3\\2&\end{smallmatrix}} = \frac{4}{3}$  , $Y_{\begin{smallmatrix}1\\2\\3\end{smallmatrix}} =$  . (5.35)

The Special Unitary Group, Birdtracks, and Applications in QCD

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