4.2 Irreducible representations of the group algebra

As already discussed in Note 4.1, a representation of a group G corresponds to an $\mathbb{F}[\mathsf{G}]$-module on the level of the group algebra. Hence, we expect that an irreducible representation corresponds to an irreducible $\mathbb{F}[\mathsf{G}]$-module — let us clarify what this means:

Definition 4.4 – Submodules and irreducible modules:
*Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$ and let $\mathcal{M}$ be an $\mathcal{A}$-module. $\mathcal{N} \subset \mathcal{M}$ is said to be an $\mathcal{A}$-*submodule *of $\mathcal{M}$ if*

$$(a, n) = an \in \mathcal{N} \qquad \text{for all } a \in \mathcal{A} \text{ and for all } n \in \mathcal{N} \tag{4.15}$$

(where $(\cdot, \cdot)$ is given through the multiplication on $\mathcal{M}$).

$\mathcal{M}$ is called an irreducible module *if its only submodules are the zero-module $\{0\}$ and $\mathcal{M}$ itself.*

4.2.1 (Left) regular representation

From Definition 3.2, we are already familiar with at least one particular group representation, namely the (left) regular representation $\mathcal{R}_{\mathsf{G}}$. The corresponding notion on the level of the group algebra is the (left) regular $\mathbb{F}[\mathsf{G}]$-module (*c.f.* Definition 4.3) — this is an immediate consequence of Theorem 4.1. Note that the two concepts — the (left) regular representation $\mathcal{R}_{\mathsf{G}}$ and the (left) regular $\mathbb{F}[\mathsf{G}]$-module — are completely equivalent in that they contain the same amount of information of the group and its representations.

In section 3.1, we stated (without proof!) that the regular representation of a group G, $\mathcal{R}_{\mathsf{G}}$, can be written as the direct sum of all irreducible representations of G (each irrep φ_i occurs with a weight $\dim(\varphi_i)$), *c.f.* Theorem 3.2. The same statement also holds on the level of the group algebra (without proof):

Theorem 4.2 – (Left) regular module is a sum of irreducible submodules:
Let $\mathcal{A}$ be an algebra and let $\mathcal{R}_{\mathcal{A}}$ be its (left) regular module. Then $\mathcal{R}_{\mathcal{A}}$ can be written as a direct sum of all irreducible submodules of $\mathcal{A}$, where each summond is weighted by its dimension (as a vector space).

Definition 4.6 – Idempotents and quasi-idempotents:
An operator e is said to be idempotent if it satisfies $e \cdot e = e$ (where e denotes the appropriate multiplication of such an operator). An operator $\tilde{e}$ is said to be quasi-idempotent if it satisfies $\tilde{e} \cdot \tilde{e} = \lambda \tilde{e}$ for some scalar quantity λ.

An idempotent operator (or simply an idempotent) is also referred to as a projection operator, the latter being used mostly in the physics literature, and the former in the math literature. We will use both names interchangeably in this course.

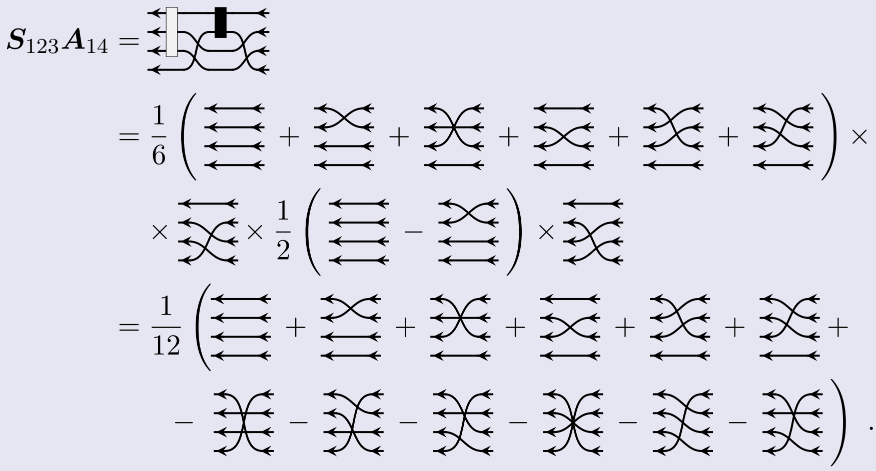
Example 4.1: Symmetrizers and antisymmetrizers
We are already familiar with two kinds of idempotent operators, namely symmetrizers and antisymmetrizers: We have shown that any symmetrizer $\boldsymbol{S}_{a_1 a_2 \ldots a_k}$ and any antisymmetrizer $\boldsymbol{A}_{a_1 a_2 \ldots a_k}$ is idempotent in Proposition 2.1.

Exercise 4.2: *Using birdtrack notation, show that the operator $\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ acting on $V^{\otimes 4}$ is quasi-idempotent: Do this by first writing the operator as a sum of permutations, and then form the product $\boldsymbol{S}_{123}\boldsymbol{A}_{14} \cdot \boldsymbol{S}_{123}\boldsymbol{A}_{14}$. Which constant α is needed to make $\alpha\boldsymbol{S}_{123}\boldsymbol{A}_{14}$*

[3] Note that $\mathscr{J}$ must be a subset of $\mathscr{I}$ to generate $\mathscr{J}$: By eq (4.18d), $ay \in \mathscr{I}$ for every $a \in \mathcal{A}$ and every $y \in \mathscr{J}$. Since the identity $\mathrm{id}_{\mathcal{A}}$ is an element of $\mathcal{A}$, it follows that $\mathrm{id}_{\mathcal{A}} y = y \in \mathscr{I}$.

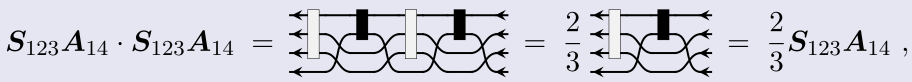
idempotent?

Solution: There are two ways of tackling this problem, the brute force method and the more elegant one. Let us discuss the brute force method first: The operator $\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ in birdtrack notation is given by



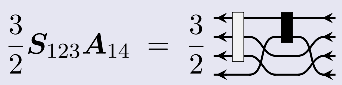
(4.19)

Multiplying the operator 4.19 by itself, we find that



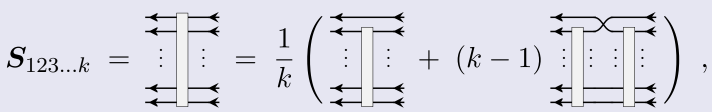
(4.20)

showing that $\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ is indeed quasi-idempotent. The necessary constant α needed to make it idempotent is $\alpha = \frac{3}{2}$; that is, the operator



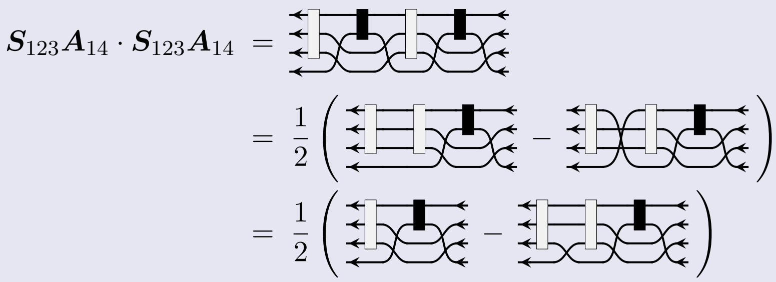
(4.21)

is idempotent.
On the other hand, one may show that $\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ is quasi-idempotent by using the formula



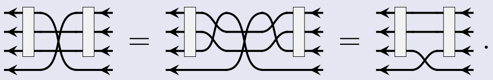
(4.22)

c.f. Proposition 2.2. Consider the product $\boldsymbol{S}_{123}\boldsymbol{A}_{14} \cdot \boldsymbol{S}_{123}\boldsymbol{A}_{14}$ and write the middle antisymmetrizer $\boldsymbol{A}_{14}$ as a sum of permutations,



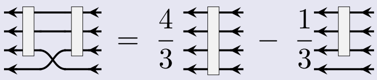
(4.23)

where we factored the appropriate permutations out of each $\boldsymbol{S}_{123}$ to the left and to the right of (14) to obtain



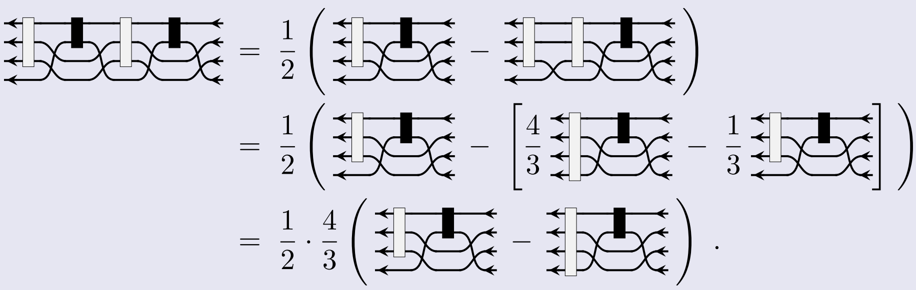
(4.24)

Using (4.22), we can express the term in eq. (4.24) as,



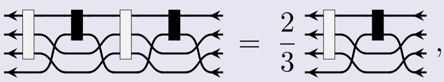
(4.25)

and substitute this back into eq. (4.23) to obtain



(4.26)

Note that the second term in eq. (4.26) vanishes since the symmetrizer $\boldsymbol{S}_{1234}$ and the anti-symmetrizer $\boldsymbol{A}_{14}$ have two legs in common (*c.f.* Proposition 2.1). Thus, we once again find that



(4.27)

showing that $\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ is quasi-idempotent and $\frac{3}{2}\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ is idempotent.

It is no coincidence that the operator $\frac{3}{2}\boldsymbol{S}_{123}\boldsymbol{A}_{14}$ in exercise 4.2 is idempotent: In fact, this operator is called a *Young projection operator*, as we will discuss in the later section 5.

Definition 4.7 – Orthogonal and primitive idempotents:
Let $\mathcal{A}$ be an algebra over a field $\mathbb{F}$, and let $\{e_i\}_1^k := \{e_1, e_2, \ldots, e_k\}$ be a subset of $\mathcal{A}$. If

$$e_i^2 = e_i \qquad \forall e_i \in \{e_i\}_1^k \tag{4.28a}$$

$$e_i e_j = \delta_{ij} \qquad \forall e_i, e_j \in \{e_i\}_1^k , \tag{4.28b}$$

then the set $\{e_i\}_1^k$ is said to form a set of orthogonal idempotents of $\mathcal{A}$.

Let $e \in \mathcal{A}$ be an idempotent element. We say that e is primitive if there exist no two orthogonal idempotents $e_1, e_2 \in \mathcal{A}$ such that $e = e_1 + e_2$.

The Special Unitary Group, Birdtracks, and Applications in QCD

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