3 Representations of a group

Much of the material of this section follows the presentation given in [7]

Definition 3.1 – Representation of a group:
Let G *be a group. A* representation φ *of* G *is a homomorphism from* G *to the endomorphism group of a vector space* V *over a field* $\mathbb{F}$*.*

$$\varphi : \mathsf{G} \longrightarrow End(V) \ . \tag{3.1}$$

The vector space V *is said to carry the representation* φ *of* G*, and is sometimes also referred to as the* carrier space *of the representation* φ*. We refer to the dimension of the carrier space* $dim(V)$ *as the* dimension of the representation φ*.*

If one wishes to make the carrier space explicit, one also commonly refers to the tuple (φ, V) *as representation of* G*.*

Note that for φ to be a homomorphism, it needs to satisfy for all $\mathsf{g}, \mathsf{h} \in \mathsf{G}$

$$\varphi(\mathsf{gh}) = \varphi(\mathsf{g})\varphi(\mathsf{h}) \tag{3.2a}$$
$$\varphi(\mathrm{id}_\mathsf{G}) = \mathbb{1}_V \ , \tag{3.2b}$$

where id_G is the identity of G and $\mathbb{1}_V$ is the identity in $\mathrm{End}_\mathbb{F}(V)$.

Example 3.1: Representation of S_3 on $\mathbb{R}^3$
For the group S_3, one can define a map $\varphi : S_3 \to \mathbb{R}^3$ as

$$\begin{aligned}
\varphi(\mathrm{id}_3) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , & \varphi((12)) &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \\
\varphi((123)) &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , & \varphi((23)) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \\
\varphi((132)) &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , & \varphi((23)) &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{3.3}$$

To see that this map defines a representation of S_n on $\mathbb{R}^3$, we need to check whether it is a group homomorphism: Clearly, the identity id_3 gets mapped to the identity in $\mathbb{R}^3$, and by direct calculation it can be verified that property (3.2a) is satisfied as well.

Let φ be a representation of a group G, $\varphi : \mathsf{G} \to \mathrm{End}(V)$. Note that, by eqns. (3.2), we have for every $\mathsf{g} \in \mathsf{G}$

$$\varphi(\mathsf{g})\varphi(\mathsf{g}^{-1}) = \varphi(\mathsf{g}\mathsf{g}^{-1}) = \varphi(\mathrm{id}_\mathsf{G}) = \mathbb{1}_v \ , \qquad \text{implying that} \quad \varphi(\mathsf{g}^{-1}) = [\varphi(\mathsf{g})]^{-1} \ . \tag{3.4}$$

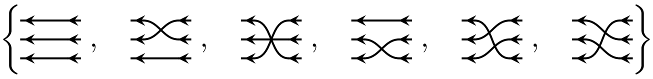
Thus, for every $\mathsf{g} \in \mathsf{G}$, $\varphi(\mathsf{g}^{-1}) \in \mathrm{End}(V)$ is the inverse map of $\varphi(\mathsf{g}) \in \mathrm{End}(V)$.

3.1 (Left) regular representation

A particular representation that will turn out to be useful is the left regular representation:

Let G be a (finite) group and let $\widehat{\mathsf{G}}$ denote the set of all elements of G in a particular order. For example, if $\mathsf{G} = S_3$, we may impose the following order to obtain

$$\widehat{S_3} := \{ \ldots \} . \tag{3.6}$$



■ **Definition 3.2:**
(Left) regular representation of a group The left action of G *on* $\widehat{\mathsf{G}}$ *defines a representation* $\mathcal{R}$ *of* G *to the* $|\mathsf{G}| \times |\mathsf{G}|$ *matrices,*

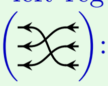
$$\mathcal{R} : \mathsf{G} \times \widehat{\mathsf{G}} \to \mathsf{GL}(\mathbb{C}, |\mathsf{G}|) \ , \tag{3.7}$$

where, for each $\mathsf{g} \in \mathsf{G}$*, the* (i,j)*-entry of the matrix* $\mathcal{R}(\mathsf{g})$ *is*

$$(i,j)\text{-}entry \quad \longrightarrow \quad \begin{cases} 1 & \text{if } \mathsf{g}_i = \mathsf{g}\mathsf{g}_j \qquad (\mathsf{g}_i \text{ is the } i^{th} \text{ entry in } \widehat{\mathsf{G}}) \\ 0 & \text{otherwise} . \end{cases} \tag{3.8}$$

The map $\mathcal{R}$ *is called the* left regular representation *of the group* G*, and it has dimension* $|\mathsf{G}|$*.*

Example 3.2: Left regular representation of S_3
As an example, consider the symmetric group S_3, and let the partially ordered set $\widehat{S_3}$ be as given in eq. (3.6). Let $\mathcal{R}$ be the left regular representation of S_3 onto $\mathsf{GL}(\mathbb{C}, 3!)$. Let us compute the matrix $\mathcal{R}((123)) = \mathcal{R}\left(\ldots \right)$:



For each $g_i \in \widehat{S_3}$, we have that

$$
\begin{aligned}
\cdot \quad = \quad & \iff & g_2 = (123)g_1 & \implies & (2,1)\text{-entry of } \mathcal{R}\left((123)\right) \text{ is } 1 \\
\cdot \quad = \quad & \iff & g_3 = (123)g_2 & \implies & (3,2)\text{-entry of } \mathcal{R}\left((123)\right) \text{ is } 1 \\
\cdot \quad = \quad & \iff & g_1 = (123)g_3 & \implies & (1,3)\text{-entry of } \mathcal{R}\left((123)\right) \text{ is } 1 \\
\cdot \quad = \quad & \iff & g_5 = (123)g_4 & \implies & (5,4)\text{-entry of } \mathcal{R}\left((123)\right) \text{ is } 1 \\
\cdot \quad = \quad & \iff & g_6 = (123)g_5 & \implies & (6,5)\text{-entry of } \mathcal{R}\left((123)\right) \text{ is } 1 \\
\cdot \quad = \quad & \iff & g_4 = (123)g_6 & \implies & (4,6)\text{-entry of } \mathcal{R}\left((123)\right) \text{ is } 1
\end{aligned}
\tag{3.9}
$$

The calculation (3.9) gives all non-zero entries of the matrix $\mathcal{R}\left((123)\right)$. Thus, $\mathcal{R}\left((123)\right)$ is given by

$$
\mathcal{R}\left((123)\right) = \mathcal{R}\left(\quad \right) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{3.10}
$$

Exercise 3.1: *Consider the symmetric group S_3 and let $\mathcal{R} : S_3 \times \widehat{S_3} \to \mathsf{GL}(\mathbb{C}, 3!)$ denote its left regular representation. Calculate the matrices $\mathcal{R}\left(id_3\right)$, $\mathcal{R}\left((123)\right)$, $\mathcal{R}\left((132)\right)$, $\mathcal{R}\left((12)\right)$, $\mathcal{R}\left((13)\right)$ and $\mathcal{R}\left((23)\right)$*

Solution:

$$
\mathcal{R}\left(\quad \right) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \mathcal{R}\left(\quad \right) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.11a}
$$

$$
\mathcal{R}\left(\quad \right) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} , \qquad \mathcal{R}\left(\quad \right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \tag{3.11b}
$$

$$\mathcal{R}\left(\quad\right) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad \mathcal{R}\left(\quad\right) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.11c}$$

The Special Unitary Group, Birdtracks, and Applications in QCD

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