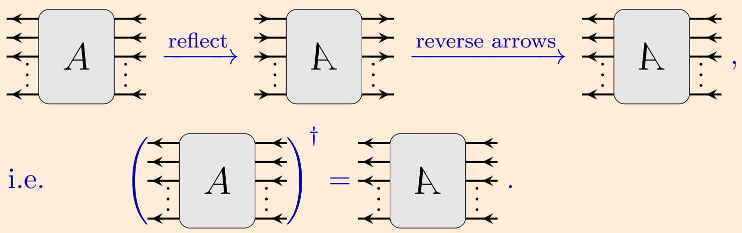
2.2.2 Hermitian conjugate of a birdtrack

Note 2.2: Hermitian Conjugate of birdtracks

Let A be a birdtrack operator. Its Hermitian conjugate with respect to the scalar product (2.13) in the birdtrack formalism is formed by flipping the birdtrack about the vertical axis and reversing the arrows[a]; for example,



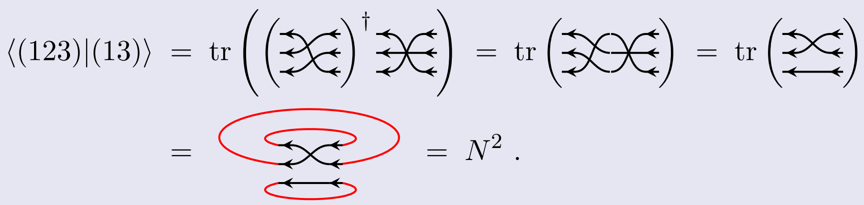
(2.18)

[a]Again, we have not specified the space on which A acts as this procedure is true in general, irrespective of the space.

> **Important:** Pay close attention to the differences between the procedures described in Note 2.2 and in Note 1.2: In Note 2.2, we explained that the Hermitian conjugate of *any* birdtrack operator is formed via reflecting the birdtrack about its vertical axis and reversing the arrows. In comparison, Note 1.2 that one obtains the inverse *only of an element of of S_n* via reflecting and reversing arrows — the procedure for taking the Hermitian conjugate is valid *for all* birdtrack operators, while the procedure for taking the inverse holds *only for the elements of S_n*!

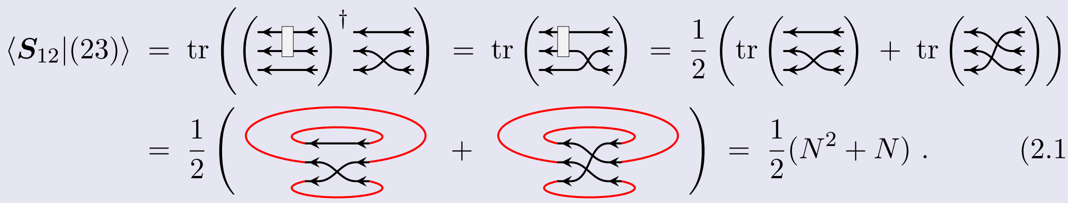
Exercise 2.6: *Calculate the following scalar products in the birdtrack formalism: $\langle(123)|(13)\rangle$ in S_3, $\langle \boldsymbol{S}_{12}|(23)\rangle$ in S_3, $\langle(234)|(13)(24)\rangle$ in S_4.*

Solution: We have that



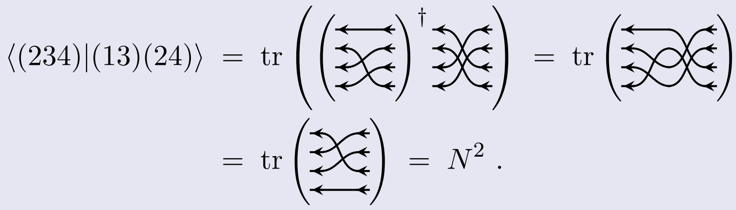
(2.19a)

Furthermore,



(2.19b)

Lastly,



(2.19c)

However, being clear about the different procedures, we immediately arrive at the following result for the elements of S_n

Corollary 2.1 – Unitarity and Hermiticity of the elements of S_n:
Every single element of S_n is unitary, that is

$$\rho^{-1} = \rho^{\dagger} , \qquad \text{for all } \rho \in S_n . \tag{2.20}$$

Furthermore, the elements of S_n are Hermitian if and only if its corresponding birdtrack is symmetric under a flip about its vertical axis.[1]

Another immediate corollary of Note 2.2 is:

[1] Calling an element of S_n an *involution* if it is its own inverse, we see that every involution in S_n is Hermitian.

■ **Corollary 2.2 – Mirror-symmetric birdtracks:**
Let A be a birdtrack operator. If A remains unchanged under a flip about its vertical axis (i.e. A is mirror-symmetric about its vertical axis) then A is Hermitian with respect to the scalar product (2.13).

Important: The converse statement of Corollary 2.2, namely that a birdtrack that is not mirror-symmmetric about its vertical axis is not Hermitian, is *not* true in general! In fact, at a later stage in this course, we will see explicit examples of non-mirror-symmetric operators that turn out to be Hermitian.

If a *Hermitian* projection operator A projects onto a subspace completely contained in the image of a Hermitian projection operator B, then we denote this as $A \subset B$, transferring the familiar notation of sets to the associated projection operators. In particular, $A \subset B$ if and only if

$$A \cdot B = B \cdot A = A \tag{2.21}$$

for the following reason: If the subspaces obtained by the consecutive application of the operators A and B in any order is the same as that obtained by merely applying A, then not only need the subspaces onto which A and B project overlap (as otherwise $A \cdot B = B \cdot A = 0$), but the subspace corresponding to A must be completely contained in the subspace of B — otherwise the last equality of (2.21) would not hold. Notice that Hermiticity is crucial for these statements — it does not apply to a general non-Hermitian operator.

A by now familiar example for this situation is the relation between (anti-) symmetrizers of different length: a symmetrizer $\boldsymbol{S}_{\mathcal{N}}$ can be absorbed into a symmetrizer $\boldsymbol{S}_{\mathcal{N}'}$, as long as the index set $\mathcal{N}$ is a subset of $\mathcal{N}'$, and the same statement holds for antisymmetrizer, [1],

$$\boldsymbol{S}_{\mathcal{N}}\boldsymbol{S}_{\mathcal{N}'} = \boldsymbol{S}_{\mathcal{N}'} = \boldsymbol{S}_{\mathcal{N}'}\boldsymbol{S}_{\mathcal{N}} \qquad \text{and} \qquad \boldsymbol{A}_{\mathcal{N}}\boldsymbol{A}_{\mathcal{N}'} = \boldsymbol{A}_{\mathcal{N}'} = \boldsymbol{A}_{\mathcal{N}'}\boldsymbol{A}_{\mathcal{N}} \ ; \tag{2.22a}$$

this can be proven in a similar way as Proposition 2.1 and is therefore left as an exercise to the reader. What eq. (2.22a) tells us is that the image of $\boldsymbol{S}_{\mathcal{N}'}$ is contained in the image of $\boldsymbol{S}_{\mathcal{N}}$, $\mathrm{im}(\boldsymbol{S}_{\mathcal{N}'}) \subset \mathrm{im}(\boldsymbol{S}_{\mathcal{N}})$, and similarly for the images of $\boldsymbol{A}_{\mathcal{N}'}$ and $\boldsymbol{A}_{\mathcal{N}}$. In a slight abuse of notation we transfer the inclusion of images to the operators, saying that

$$\boldsymbol{S}_{\mathcal{N}'} \subset \boldsymbol{S}_{\mathcal{N}} \qquad \text{and} \qquad \boldsymbol{A}_{\mathcal{N}'} \subset \boldsymbol{A}_{\mathcal{N}} \qquad \text{whenever } \mathcal{N} \subset \mathcal{N}' \ . \tag{2.22b}$$

Example 2.2:
Considering the symmetrizers $\boldsymbol{S}_{123}$ and $\boldsymbol{S}_{12}$, we have

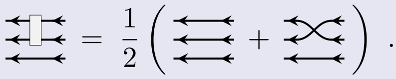
$$= \qquad = \qquad ; \tag{2.23a}$$

we can think of the "smaller" symmetrizer (over less index kegs) as being absorbed by the larger one. Thus, by the above notation, $\boldsymbol{S}_{123} \subset \boldsymbol{S}_{12}$,

$$\subset \qquad . \tag{2.23b}$$

Exercise 2.7: *Show explicitly that eq. (2.23a) holds.*

Solution: By definition, we may write the symmetrizer $\boldsymbol{S}_{12}$ as a sum of permutations ass

$\boldsymbol{S}_{12} =$  (2.24)

Acting either of the permutations in the sum (2.24) on $\boldsymbol{S}_{123}$ merely effects a reordering of the underlying sum of $\boldsymbol{S}_{123}$, but nothing else, such that

 (2.25)

Thus, it immediately follows that eq. (2.23a) must also hold, as required.

The Special Unitary Group, Birdtracks, and Applications in QCD

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