## 2.2 Linear maps, scalar product and Hermiticity

We have already defined a product on the birdtracks of the group $S_n$ by merely connecting the index lines, *c.f.* Note 1.1. We may also define a scalar product on the space of linear maps as follows:

■ **Definition 2.5 – Scalar product:**
*Let* $A, B$ *be linear maps from* $V^{\otimes k}$ *to itself, that is* $A, B \in \mathrm{Lin}\left(V^{\otimes k}\right)$. *We define a scalar product* $\langle\cdot|\cdot\rangle$ *between these maps as*

$$\langle A|B\rangle := tr\left(A^{\dagger}B\right) \ , \tag{2.13}$$

*where* $A^{\dagger}$ *denotes the Hermitian conjugate (i.e. complex conjugate transpose) of* $A$.

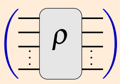
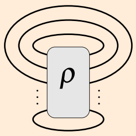
Unless explicitly stated otherwise, we will from now on always assume the product (2.13) whenever reference to a scalar product is required.

To apply the product (2.13) to operators in the birdtrack formalism, we first need to be able to form the Hermitian conjugate and take a trace in the birdtack formalism. Let us start with the latter:

### 2.2.1 Trace of a birdtrack

**Note 2.1: Tracing birdtracks**

Let $\rho$ be a birdtrack operator. Its trace $\mathrm{tr}(\rho)$ is formed by connecting the index lines on the same level,

tr  :=  , (2.14)

and replacing each closed loop by a factor $\dim(V) = N$ (note that loops may self-intersect).[a]

[a] Note that no reference has been made with respect to the space on which $\rho$ operates, and no arrows have been added to the birdtracks in eq. (2.14). The reason for this is that eq. (2.14) does not only define the trace of operators on $V^{\otimes k}$, but also on more general product spaces, where each closed loop is replaced by the dimension of the space on which it acts.

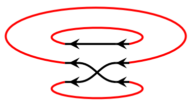
You may wonder why the procedure described in Note 2.1 indeed yields the trace of a birdtrack operator $\rho$. Let us motivate this by once again looking at the elements of $S_n$: When writing these elements as products of Kronecker $\delta$'s as described in eqns. (1.18) (just after Note 1.1), the trace is formed by a contraction of indices, for example,

$$\mathrm{tr}\left( \quad \right) = \mathrm{tr}\left(\delta^{b_1}{}_{a_1}\delta^{b_3}{}_{a_2}\delta^{b_2}{}_{a_3}\right) \xrightarrow[\text{contraction}]{\text{index}} \delta^{b_1}{}_{b_1}\delta^{b_3}{}_{b_2}\delta^{b_2}{}_{b_3} \tag{2.15a}$$

But how does one contract indices as indicated in eq. (2.15a)? By means of a multiplication with another Kronecker $\delta$, such that

$$\mathrm{tr}\left(\delta^{b_1}{}_{a_1}\delta^{b_3}{}_{a_2}\delta^{b_2}{}_{a_3}\right) = \left(\delta^{b_1}{}_{a_1}\delta^{a_1}{}_{b_1}\right)\left(\delta^{b_3}{}_{a_2}\delta^{a_2}{}_{b_2}\right)\left(\delta^{b_2}{}_{a_3}\delta^{a_3}{}_{b_3}\right) = \delta^{b_1}{}_{b_1}\delta^{b_3}{}_{b_2}\delta^{b_2}{}_{b_3} \;, \tag{2.15b}$$

where we have written the Kronecker $\delta$'s arising from the trace operation in red for visual clarity. In birdtrack notation, however, we merely denote a Kronecker $\delta$ by a line (*c.f.* eqns. (1.18)), such that

$$\left(\delta^{b_1}{}_{a_1}\delta^{a_1}{}_{b_1}\right)\left(\delta^{b_3}{}_{a_2}\delta^{a_2}{}_{b_2}\right)\left(\delta^{b_2}{}_{a_3}\delta^{a_3}{}_{b_3}\right) =$$  , (2.15c)

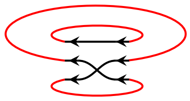
where the lines corresponding to the red Kronecker $\delta$'s were also drawn red. Furthermore, if, as in our case, $\delta^b{}_a : V \to V$ with $\dim(V) = N$, then it immediately follows that

$$\mathrm{tr}\left(\delta^b{}_a\right) = \delta^a{}_a = \dim(V) = N \;. \tag{2.16a}$$

Hence, in the example (2.15b), we have that

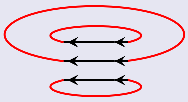
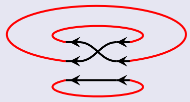
$$\mathrm{tr}\left(\delta^{b_1}{}_{a_1}\delta^{b_3}{}_{a_2}\delta^{b_2}{}_{a_3}\right) = \delta^{b_1}{}_{b_1}\delta^{b_3}{}_{b_2}\delta^{b_2}{}_{b_3} = \delta^{b_1}{}_{b_1}\delta^{b_3}{}_{b_3} = N^2 \;. \tag{2.16b}$$

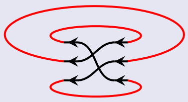
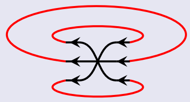
Notice that we used the fact that $\delta^{a_3}{}_{a_2}\delta^{a_2}{}_{a_3} = \delta^{a_3}{}_{a_3}$, two Kronecker $\delta$'s combined into one as their indices were contracted. Graphically, this corresponds to two Kronecker $\delta$ lines being connected (at the point representing the contracted index). This observations warrants the statement that each closed loop (even if it self-intersects!) of a birdtrack gives rise to a factor $N$,

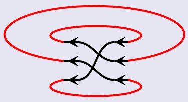
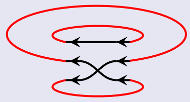
$\mathrm{tr}\left(\delta^{b_1}{}_{a_1}\delta^{b_3}{}_{a_2}\delta^{b_2}{}_{a_3}\right) = \mathrm{tr}\left(\quad\right) =$  $= N^2$ . (2.16c)

**Exercise 2.5:** *Find the trace of all elements in $S_3$ on $V^{\otimes 3}$.*

**Solution:** Drawing the lines originating from the trace in red for visual clarity, we have

$\mathrm{tr}\left(\quad\right) =$  $= N^3$ , $\mathrm{tr}\left(\quad\right) =$  $= N^2$ ,

$\mathrm{tr}\left(\quad\right) =$  $= N$ , $\mathrm{tr}\left(\quad\right) =$  $= N^2$ , (2.17)

$\mathrm{tr}\left(\quad\right) =$  $= N$ , $\mathrm{tr}\left(\quad\right) =$  $= N^2$ .

Let us now move on to the Hermitian conjugate of birdtracks:

# The Special Unitary Group, Birdtracks, and Applications in QCD

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