group theory - week 9

Continuous groups

Georgia Tech PHYS-7143

Homework HW9 due Tuesday 2019-03-12

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 9.1 *Irreps of* $SO(2)$	2 points
Exercise 9.2 *Reduction of product of two* $SO(2)$ *irreps*	1 point
Exercise 9.3 *Irreps of* $O(2)$	2 points
Exercise 9.4 *Reduction of product of two* $O(2)$ *irreps*	1 point
Exercise 9.5 *A fluttering flame front*	4 points

Bonus points
Exercise 9.6 $O(2)$ *fundamental domain for Kuramoto-Sivashinsky equation* (difficult) 10 points

Total of 10 points = 100 % score.

2019-03-05 Predrag Lecture 17 Continuous groups

This lecture is not taken from any particular book, it's about basic ideas of how one goes from finite groups to the continuous ones that any physicist should know. The main idea comes from discrete groups. We have worked one example out in week 2, the discrete Fourier transform of example 2.5 *Projection operators for cyclic group C_N*. The cyclic group C_N is generated by the powers of the rotation by $2\pi/N$, and in general, in the $N \to \infty$ limit one only needs to understand the algebra of T_ℓ, generators of infinitesimal transformations, $D(\theta) = 1 + i\sum_\ell \theta_\ell T_\ell$. Applied to functions, they turn out to be partial derivatives.

Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sect. 6.6 completes discussion of Fourier analysis as continuum limit of cyclic groups C_n, compares SO(2), O(2), discrete translations group, and continuous translations group.

2019-03-07 Predrag Lecture 18 Lie groups. Matrix representations

The $N \to \infty$ limit of C_N gets you to the continuous Fourier transform as a representation of $\mathrm{U}(1) \simeq \mathrm{SO}(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. So we need a fresh restart; that is afforded by matrix groups, and in particular the unitary group $\mathrm{U}(n) = \mathrm{U}(1) \otimes \mathrm{SU}(n)$, which contains all other compact groups, finite or continuous, as subgroups.

Reading: Chen, Ping and Wang [7] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples* (click here).

Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sects. 6.1-6.3 Irreps of SO(2). In particular, note that while geometrically intuitive representation is the set of rotation $[2\times 2]$ matrices, they split into pairs of 1-dimensional irreps.

Sect. 9.1 that follows is a very condensed extract of chapters 3 *Invariants and reducibility* and 4 *Diagrammatic notation* from *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [9]. I am usually reluctant to use birdtrack notations in front of graduate students indoctrinated by their professors in the 1890's tensor notation, but today I'm emboldened by the very enjoyable article on *The new language of mathematics* by Dan Silver [17]. Your professor's notation is as convenient for actual calculations as -let's say- long division using roman numerals. So leave them wallowing in their early progressive rock of 1968, King Crimsons of their youth. You chill to beats younger than Windows 98, to grime, to trap, to hardvapour, to birdtracks.

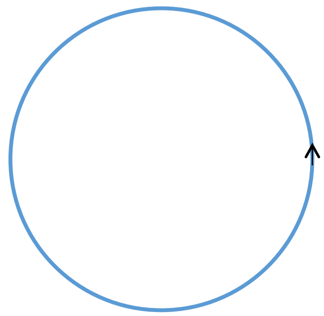


Figure 9.1: Circle group $S^1 = \mathrm{SO}(2)$, the symmetry group of a circle with directed rotations, is a compact group, as its natural parametrization is either the angle $\phi \in$ [0, 2ß), or the perimeter $x \in [0, L)$.

9.1 Lie groups for pedestrians

> [...] which is an expression of consecration of angular momentum.
>
> — Mason A. Porter's student

Definition: A Lie group is a topological group G such that (i) G has the structure of a smooth differential manifold, and (ii) the composition map $G \times G \to G : (g, h) \to gh^{-1}$ is smooth, i.e., $\mathbb{C}^\infty$ differentiable.

Do not be mystified by this definition. Mathematicians also have to make a living. The compact Lie groups that we will deploy here are a generalization of the theory of $\mathrm{SO}(2) \simeq \mathrm{U}(1)$ rotations, i.e., Fourier analysis. By a 'smooth differential manifold' one means objects like the circle of angles that parameterize continuous rotations in a plane, figure 9.1, or the manifold swept by the three Euler angles that parameterize $\mathrm{SO}(3)$ rotations.

By 'compact' one means that these parameters run over finite ranges, as opposed to parameters in hyperbolic geometries, such as Minkowsky $\mathrm{SO}(3, 1)$. The groups we focus on here are compact by default, as their representations are linear, finite-dimensional matrix subgroups of the unitary matrix group $\mathrm{U}(d)$.

Example 1. Circle group. A circle with a direction, figure 9.1, is invariant under rotation by any angle $\theta \in [0, 2\pi)$, and the group multiplication corresponds to composition of two rotations $\theta_1 + \theta_2 \mod 2\pi$. The natural representation of the group action is by a complex numbers of absolute value 1, i.e., the exponential $e^{i\theta}$. The composition rule is then the complex multiplication $e^{i\theta_2} e^{i\theta_1} = e^{i(\theta_1+\theta_2)}$. The circle group is a *continuous group*, with infinite number of elements, parametrized by the continuous parameter $\theta \in [0, 2\pi)$. It can be thought of as the $n \to \infty$ limit of the cyclic group C_n. Note that the circle divided into n segments is *compact*, in distinction to the infinite lattice of integers $\mathbb{Z}$, whose limit is a *line* (noncompact, of infinite length).

An element of a $[d\times d]$ -dimensional matrix representation of a *Lie group* continuously connected to identity can be written as

$$g(\boldsymbol{\phi}) = e^{i\boldsymbol{\phi}\cdot T}\,, \qquad \boldsymbol{\phi}\cdot T = \sum_{a=1}^{N}\phi_a T_a\,, \tag{9.1}$$

where $\boldsymbol{\phi}\cdot T$ is a *Lie algebra* element, T_a are matrices called 'generators', and $\boldsymbol{\phi} = (\phi_1, \phi_2, \cdots, \phi_N)$ are the parameters of the transformation. Repeated indices are summed throughout, and the dot product refers to a sum over Lie algebra generators. Sometimes it is convenient to use the Dirac bra-ket notation for the Euclidean product of two real vectors $x, y \in \mathbb{R}^d$, or the product of two complex vectors $x, y \in \mathbb{C}^d$, i.e., indicate complex x-transpose times y by

$$\langle x|y\rangle = x^\dagger y = \sum_i^d x_i^* y_i\,. \tag{9.2}$$

Finite unitary transformations $\exp(i\boldsymbol{\phi}\cdot T)$ are generated by sequences of infinitesimal steps of form

$$g(\boldsymbol{\delta\phi}) \simeq 1 + i\boldsymbol{\delta\phi}\cdot T\,, \quad \boldsymbol{\delta\phi}\in\mathbb{R}^N\,, \quad |\boldsymbol{\delta\phi}| \ll 1\,, \tag{9.3}$$

where T_a, the *generators* of infinitesimal transformations, are a set of linearly independent $[d\times d]$ hermitian matrices (see figure 9.2 (b)).

The reason why one can piece a global transformation from infinitesimal steps is that the choice of the "origin" in coordinatization of the group manifold sketched in figure 9.2 (a) is arbitrary. The coordinatization of the tangent space at one point on the group manifold suffices to have it everywhere, by a coordinate transformation g, i.e., the new origin y is related to the old origin x by conjugation $y = g^{-1}xg$, so all tangent spaces belong the same class, they are geometrically equivalent.

Unitary and orthogonal groups are defined as groups that preserve 'length' norms, $\langle gx|gx\rangle = \langle x|x\rangle$, and infinitesimally their generators (9.3) induce no change in the norm, $\langle T_a x|x\rangle + \langle x|T_a x\rangle = 0$, hence the Lie algebra generators T_a are hermitian for,

$$T_a^\dagger = T_a\,. \tag{9.4}$$

The flow field at the state space point x induced by the action of the group is given by the set of N tangent fields

$$t_a(x)_i = (T_a)_{ij}x_j\,, \tag{9.5}$$

which span the d-dimensional *group tangent space* at state space point x, parametrized by $\boldsymbol{\delta\phi}$.

For continuous groups the Lie algebra, i.e., the algebra spanned by the set of N generators T_a of infinitesimal transformations, takes the role that the $|G|$ group elements play in the theory of discrete groups (see figure 9.2).

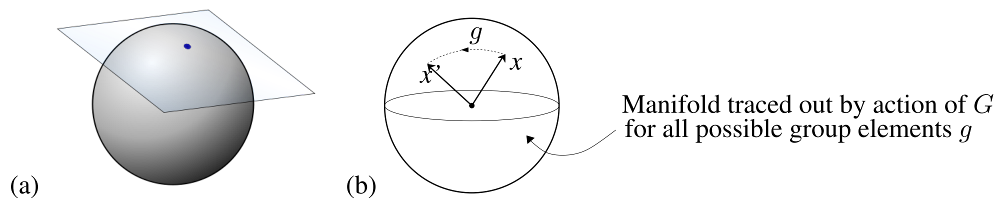


Figure 9.2: (a) Lie algebra fields $\{t_1, \cdots, t_N\}$ span the tangent space of the group orbit $\mathcal{M}_x$ at state space point x, see (9.5) (figure from WikiMedia.org). (b) A global group transformation $g : x \to x'$ can be pieced together from a series of infinitesimal steps along a continuous trajectory connecting the two points. The group orbit of state space point $x \in \mathbb{R}^d$ is the N-dimensional manifold of all actions of the elements of group G on x.

9.1.1 Invariants

One constructs the irreps of finite groups by identifying matrices that commute with all group elements, and using their eigenvalues to decompose arbitrary representation of the group into a unique sum of irreps. The same strategy works for the compact Lie groups, (9.9), and is indeed the key idea that distinguishes the invariance groups classification developed in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [9] from the 19th century Cartan-Killing classification of Lie algebras.

Definition. The vector $q \in V$ is an *invariant vector* if for any transformation $g \in \mathcal{G}$

$$q = Gq\,. \tag{9.6}$$

Definition. A tensor $x \in V^p \otimes \bar{V}^q$ is an *invariant tensor* if for any $g \in G$

$$x^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q} = G^{a_1}{}_{c_1} G^{a_2}{}_{c_2} \ldots G_{b_1}{}^{d_1} \ldots G_{b_q}{}^{d_q} x^{c_1 c_2 \ldots c_p}_{d_1 \ldots d_q}\,. \tag{9.7}$$

If a bilinear form $m(\bar{x}, y) = x^a M_a{}^b y_b$ is invariant for all $g \in \mathcal{G}$, the matrix

$$M_a{}^b = G_a{}^c G^b{}_d M_c{}^d \tag{9.8}$$

is an *invariant matrix*. Multiplying with $G_b{}^e$ and using the unitary, we find that the invariant matrices *commute* with all transformations $g \in \mathcal{G}$:

$$[G, \mathbf{M}] = 0\,. \tag{9.9}$$

Definition. An *invariance group* $\mathcal{G}$ is the set of all linear transformations (9.7) that preserve the primitive invariant relations (and, by extension, *all* invariant relations)

$$\begin{aligned} p_1(x, \bar{y}) &= p_1(Gx, \bar{y}G^\dagger) \\ p_2(x, y, z, \ldots) &= p_2(Gx, Gy, Gz \ldots)\,, \qquad \ldots\,. \end{aligned} \tag{9.10}$$

Unitarity guarantees that all contractions of primitive invariant tensors, and hence all composed tensors $h \in H$, are also invariant under action of $\mathcal{G}$. As we assume unitary $\mathcal{G}$, it follows that the list of primitives must always include the Kronecker delta.

Example 2. If $p^a q_a$ is the only invariant of $\mathcal{G}$

$$p'^a q'_a = p^b (G^\dagger G)_b{}^c q_c = p^a q_a \,, \tag{9.11}$$

then $\mathcal{G}$ is the full *unitary group* $U(n)$ (invariance group of the complex norm $|x|^2 = x^b x_a \delta^a_b$), whose elements satisfy

$$G^\dagger G = 1 \,. \tag{9.12}$$

Example 3. If we wish the z-direction to be invariant in our 3-dimensional space, $q = (0, 0, 1)$ is an invariant vector (9.6), and the invariance group is $O(2)$, the group of all rotations in the x-y plane.

9.1.2 Discussion

Qimen Xu Please explain when one keeps track of the order of tensorial indices?

Predrag In a tensor, upper, lower indices are separately ordered - and that order matters. The simplest example: if some indices form an antisymmetric pair, writing them in wrong order gives you a wrong sign. In a matrix representation of a group action, one has to distinguish between the "in" set of indices – the ones that get contracted with the initial tensor, and the "out" set of indices that label the tensor after the transformation. Only if the matrix is Hermitian the order does not matter. If you understand Eq. (3.22) in birdtracks.eu, you get it. Does that answer your question?

9.1.3 Infinitesimal transformations, Lie algebras

A unitary transformation G infinitesimally close to unity can be written as

$$G_a{}^b = \delta^b_a + iD^b_a \,, \tag{9.13}$$

where D is a hermitian matrix with small elements, $|D^b_a| \ll 1$. The action of $g \in \mathcal{G}$ on the conjugate space is given by

$$(G^\dagger)_b{}^a = G^a{}_b = \delta^a_b - iD^a_b \,. \tag{9.14}$$

D can be parametrized by $N \le n^2$ real parameters. N, the maximal number of independent parameters, is called the *dimension* of the group (also the dimension of the Lie algebra, or the dimension of the adjoint rep).

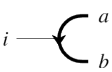
Here we shall consider only infinitesimal transformations of form $G = 1 + iD$, $|D^a_b| \ll 1$. We do not study the entire group of invariant transformation, but only the transformations connected to the identity. For example, we shall not consider invariances under coordinate reflections.

The generators of infinitesimal transformations (9.13) are hermitian matrices and belong to the $D^a_b \in V \otimes \bar{V}$ space. However, not any element of $V \otimes \bar{V}$ generates an allowed transformation; indeed, one of the main objectives of group theory is to define the class of allowed transformations.

This subspace is called the *adjoint* space, and its special role warrants introduction of special notation. We shall refer to this vector space by letter A, in distinction to the defining space V. We shall denote its dimension by N, label its tensor indices by $i, j, k \ldots$, denote the corresponding Kronecker delta by a thin, straight line,

$$\delta_{ij} = i \;\text{———}\; j\,, \qquad i, j = 1, 2, \ldots, N\,, \tag{9.15}$$

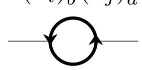
and the corresponding transformation generators by

$$(C_A)_{i,}{}^a_b = \frac{1}{\sqrt{a}}(T_i)^a_b = i$$


$$a, b = 1, 2, \ldots, n \qquad i = 1, 2, \ldots, N\,.$$

Matrices T_i are called the *generators* of infinitesimal transformations. Here a is an (uninteresting) overall normalization fixed by the orthogonality condition

$$(T_i)^a_b (T_j)^b_a = \mathrm{tr}\,(T_i T_j) = a\,\delta_{ij}$$



$$= a \;\text{———}\;. \tag{9.16}$$

For every invariant tensor q, the infinitesimal transformations $G = 1 + iD$ must satisfy the invariance condition (9.6). Parametrizing D as a projection of an arbitrary hermitian matrix $H \in V \otimes \bar{V}$ into the adjoint space, $D = \mathbf{P}_A H \in V \otimes \bar{V}$,

$$D^a_b = \frac{1}{a}(T_i)^a_b \epsilon_i\,, \tag{9.17}$$

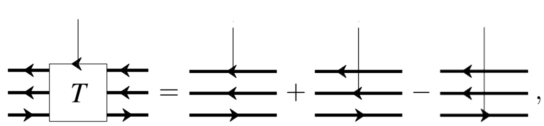
we obtain the *invariance condition* which the *generators* must satisfy: they *annihilate* invariant tensors:

$$T_i q = 0\,. \tag{9.18}$$

To state the invariance condition for an arbitrary invariant tensor, we need to define the action of generators on the tensor reps. By substituting $G = 1 + i\epsilon \cdot T + O(\epsilon^2)$ and keeping only the terms linear in ϵ, we find that the generators of infinitesimal transformations for tensor reps act by touching one index at a time:

$$\begin{aligned}
(T_i)^{a_1 a_2 \ldots a_p}_{b_1 \ldots b_q},{}^{d_q \ldots d_1}_{c_p \ldots c_2 c_1} &= (T_i)^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\
&+ \delta^{a_1}_{c_1} (T_i)^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} + \ldots + \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots (T_i)^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} \\
&- \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} (T_i)^{d_1}_{b_1} \ldots \delta^{d_q}_{b_q} - \ldots - \delta^{a_1}_{c_1} \delta^{a_2}_{c_2} \ldots \delta^{a_p}_{c_p} \delta^{d_1}_{b_1} \ldots (T_i)^{d_q}_{b_q}\,.
\end{aligned} \tag{9.19}$$

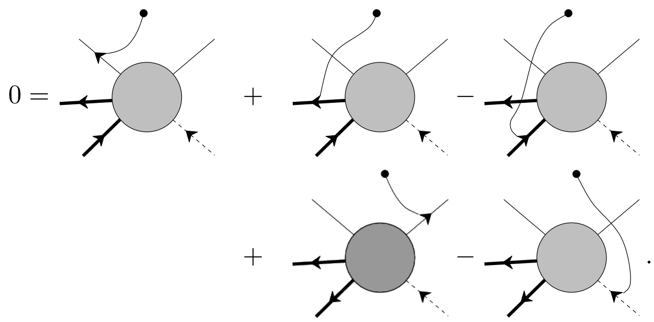
This forest of indices vanishes in the birdtrack notation, enabling us to visualize the formula for the generators of infinitesimal transformations for any tensor representation:



(9.20)

with a relative minus sign between lines flowing in opposite directions. The reader will recognize this as the Leibnitz rule.

The invariance conditions take a particularly suggestive form in the birdtrack notation. Equation (9.18) amounts to the insertion of a generator into all external legs of the diagram corresponding to the invariant tensor q:

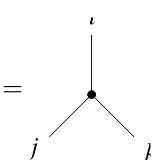


(9.21)

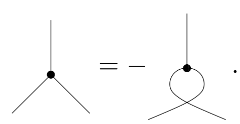
The insertions on the lines going into the diagram carry a minus sign relative to the insertions on the outgoing lines.

As the simplest example of computation of the generators of infinitesimal transformations acting on spaces other than the defining space, consider the adjoint rep. Where does the ugly word "adjoint" come from in this context is not obvious, but remember it this way: this is the one distinguished representation, which is intrinsic to the Lie algebra, with the explicit matrix elements $(T_i)_{jk}$ of the adjoint rep given by the fully antisymmetric structure constants iC_{ijk} of the algebra (i.e., its multiplication table under the commutator product). It's the continuous groups analogoue of the multiplication table, or the regular representation for the finite groups. The factor i ensures their reality (in the case of hermitian generators T_i), and we keep track of the overall signs by always reading indices *counterclockwise* around a vertex:

$$-iC_{ijk} =$$

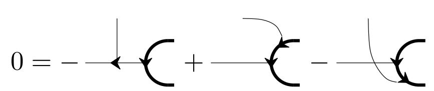


(9.22)

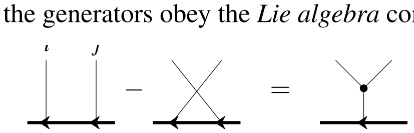


(9.23)

As all other invariant tensors, the generators must satisfy the invariance conditions (9.21):



Redrawing this a little and replacing the adjoint rep generators (9.22) by the structure constants, we find that the generators obey the *Lie algebra* commutation relation

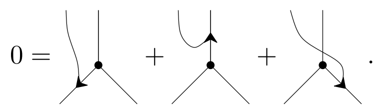


(9.24)

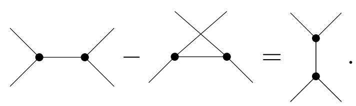
In other words, the Lie algebra commutator

$$T_i T_j \quad - \quad T_j T_i \quad = \quad i C_{ijk} T_k \,. \tag{9.25}$$

is simply a statement that T_i, the generators of invariance transformations, are themselves invariant tensors. Now, honestly, do you prefer the three-birdtracks equation (9.24), or the mathematician's page-long definition of the adjoint rep? It's a classic example of bad notation getting into way of understanding a relation of beautiful simplicity. The invariance condition for structure constants C_{ijk} is likewise



Rewriting this with the dot-vertex (9.22), we obtain



(9.26)

This is the Lie algebra commutator for the adjoint rep generators, known as the *Jacobi relation* for the structure constants

$$C_{ijm} C_{mkl} - C_{ljm} C_{mki} = C_{iml} C_{jkm} \,. \tag{9.27}$$

Hence, the Jacobi relation is also an invariance statement, this time the statement that the structure constants are invariant tensors.

9.1.4 Discussion

Lin Xin Please explain the $M_{\mu\nu,\delta\rho}$ generators of SO(n).

Predrag Let me know if you understand the derivation of Eqs. (4.51) and (4.52) in birdtracks.eu. Does that answer your question?

9.2 Birdtracks - updated history

Predrag Cvitanović November 7, 2019

Young tableaux and (non-Hermitian) Young projection operators were introduced by Young [20] in 1933 (Tung monograph [19] is a standard exposition). In 1937 R. Brauer [4] introduced diagrammatic notation for δ_{ij} in order to represent "Brauer algebra" permutations, index contractions, and matrix multiplication diagrammatically. R. Penrose's papers were the first to cast the Young projection operators into a diagrammatic form. In 1971 monograph [14] Penrose introduced diagrammatic notation for symmetrization operators, Levi-Civita tensors [16], and "strand networks" [13]. Penrose credits Aitken [2] with introducing this notation in 1939, but inspection of Aitken's book reveals a few Brauer diagrams for permutations, and no (anti)symmetrizers. Penrose's [15] 1952 initial ways of drawing symmetrizers and antisymmetrizers are very aesthetical, but the subsequent developments gave them a distinctly ostrich flavor [15]. In 1974 G. 't Hooft introduced a double-line notation for U(n) gluon group-theory weights [1]. In 1976 Cvitanović [8] introduced analogous notation for SU(N), SO(n) and Sp(n). For several specific, few-index tensor examples, diagrammatic Young projection operators were constructed by Canning [6], Mandula [12], and Stedman [18].

The 1975–2008 Cvitanović diagrammatic formulation of the theory of all semi-simple Lie groups [9] as a way to compute group theoretic wights without any recourse to symbols goes conceptually and profoundly beyond the Penrose notation (indeed, Cvitanović "birdtracks" bear no resemblance to Penrose's "fornicating ostriches" [15]).

A chapter in Cvitanović 2008 monograph [9] sketches how birdtrack (diagrammatic) Young projection operators for arbitrary irreducible representation of SU(N) could be constructed (this text is augmented by a 2005 appendix by Elvang, Cvitanović and Kennedy [10] which, however, contains a significant error). Keppeler and Sjödahl [11] systematized the construction by offering a simple method to construct Hermitian Young projection operators in the birdtrack formalism. Their iteration is easy to understand, and the proofs of Hermiticity are simple. However, in practice, the algorithm is inefficient - the expression balloon quickly, the Young projection operators soon become unwieldy and impractical, if not impossible to implement.

The Alcock-Zeilinger algorithm, based on the simplification rules of ref. [3], leads to explicitly Hermitian and drastically more compact expressions for the projection operators than the Keppeler-Sjödahl algorithm [11]. Alcock-Zeilinger fully supersedes Cvitanović's formulation, and any future full exposition of reduction of SU(N) tensor products into irreducible representations should be based on the Alcock-Zeilinger algorithm.

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Exercises

9.1. **Irreps of SO(2).** Matrix

$$T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{9.28}$$

is the generator of rotations in a plane.

(a) Use the method of projection operators to show that for rotations in the kth Fourier mode plane, the irreducible $1D$ subspaces orthonormal basis vectors are

$$\mathbf{e}^{(\pm k)} = \frac{1}{\sqrt{2}}\left(\pm\mathbf{e}_1^{(k)} - i\,\mathbf{e}_2^{(k)}\right) .$$

How does T act on $\mathbf{e}^{(\pm k)}$?

(b) What is the action of the $[2\times 2]$ rotation matrix

$$D^{(k)}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} , \qquad k = 1, 2, \cdots$$

on the $(\pm k)$th subspace $\mathbf{e}^{(\pm k)}$?

(c) What are the irreducible representations characters of $\mathrm{SO}(2)$?

9.2. **Reduction of a product of two SO(2) irreps.** Determine the Clebsch-Gordan series for $\mathrm{SO}(2)$. Hint: Abelian group has 1-dimensional characters. Or, you are just multiplying terms in Fourier series.

9.3. **Irreps of O(2).** $\mathrm{O}(2)$ is a group, but not a Lie group, as in addition to continuous transformations generated by (9.28) it has, as a group element, a parity operation

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which cannot be reached by continuous transformations.

(a) Is this group Abelian, i.e., does T commute with $R(k\theta)$? Hint: evaluate first the $[T, \sigma]$ commutator and/or show that $\sigma D^{(k)}(\theta)\sigma^{-1} = D^{(k)}(-\theta)$.

(b) What are the equivalence (i.e., conjugacy) classes of this group?

(c) What are irreps of $\mathrm{O}(2)$? What are their dimensions?

Hint: $\mathrm{O}(2)$ is the $n \to \infty$ limit of D_n, worked out in exercise 4.4 *Irreducible representations of dihedral group D_n*. Parity σ maps an $\mathrm{SO}(2)$ eigenvector into another eigenvector, rendering eigenvalues of any $\mathrm{O}(2)$ commuting operator degenerate. Or, if you really want to do it right, apply Schur's first lemma to improper rotations

$$R^{'}(\theta) = \begin{pmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{pmatrix} \sigma = \begin{pmatrix} \cos k\theta & \sin k\theta \\ \sin k\theta & -\cos k\theta \end{pmatrix}$$

to prove irreducibility for $k \neq 0$.

(d) What are irreducible characters of $\mathrm{O}(2)$?

(e) Sketch a fundamental domain for $\mathrm{O}(2)$.

9.4. **Reduction of a product of two O(2) irreps.** Determine the Clebsch-Gordan series for $\mathrm{O}(2)$, i.e., reduce the Kronecker product $D^{(k)} \bigotimes D^{(\ell)}$.

9.5. **A fluttering flame front.**

(a) Consider a linear partial differential equation for a real-valued field $u = u(x,t)$ defined on a periodic domain $u(x,t) = u(x+L,t)$:

$$u_t + u_{xx} + \nu u_{xxxx} = 0 , \qquad x \in [0, L] . \tag{9.29}$$

In this equation $t \geq 0$ is the time and x is the spatial coordinate. The subscripts x and t denote partial derivatives with respect to x and t: $u_t = \partial u/d\partial$, u_{xxxx} stands for the 4th spatial derivative of $u = u(x,t)$ at position x and time t. Consider the form of equations under coordinate shifts $x \to x + \ell$ and reflection $x \to -x$. What is the symmetry group of (9.29)?

(b) Expand $u(x,t)$ in terms of its SO(2) irreducible components (hint: Fourier expansion) and rewrite (9.29) as a set of linear ODEs for the expansion coefficients. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(c) Expand $u(x,t)$ in terms of its O(2) irreducible components (hint: Fourier expansion) and rewrite (9.29) as a set of linear ODEs. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(d) Interpret $u = u(x,t)$ as a 'flame front velocity' and add a quadratic nonlinearity to (9.29),

$$u_t + \frac{1}{2}(u^2)_x + u_{xx} + \nu u_{xxxx} = 0\,, \qquad x \in [0, L]\,. \tag{9.30}$$

This nonlinear equation is known as the Kuramoto-Sivashinsky equation, a baby cousin of Navier-Stokes. What is the symmetry group of (9.30)?

(e) Expand $u(x,t)$ in terms of its O(2) irreducible components (see exercise 9.3) and rewrite (9.30) as an infinite tower of coupled nonlinear ODEs.

(f) What are the degeneracies of the spectrum of the eigenvalues of the time evolution operator?

9.6. **O(2) fundamental domain for Kuramoto-Sivashinsky equation.** You have C_2 discrete symmetry generated by flip σ, which tiles the space by two tiles.

- Is there a subspace invariant under this C_2? What form does the tower of ODEs take in this subspace?
- How would you restrict the flow (the integration of the tower of coupled ODEs) to a fundamental domain?

This problem is indeed hard, a research level problem, at least for me and the grad students in our group. Unlike the beautiful full-reducibility, character-orthogonality representation theory of linear problems, in nonlinear problems symmetry reduction currently seems to require lots of clever steps and choices of particular coordinates, and we am not at all sure that our solution is the optimal one. Somebody looking at the problem with a fresh eye might hit upon a solution much simpler than ours. Has happened before :)

Burak Budanur's solution is written up in Budanur and Cvitanović [5] *Unstable manifolds of relative periodic orbits in the symmetry-reduced state space of the Kuramoto-Sivashinsky system* sect. 3.2 *O(2) symmetry reduction*, eq. (17) (get it here).

9.7. **Lie algebra from invariance.** Derive the Lie algebra commutator and the Jacobi identity as particular examples of the invariance condition, using both index and birdtracks notations. The invariant tensors in question are "the laws of motion," i.e., the generators of infinitesimal group transformations in the defining and the adjoint representations.