group theory - week 7

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Homework HW7 due Tuesday 2019-03-05

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 7.1 *Am I a group?*	2 points
Exercise 7.2 *Product of two groups*	2 points
Exercise 7.3 Work through ChaosBook.org example 24.2 *Unrestricted symbolic dynamics*	6 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2019-02-19 Predrag Lecture 13 Fundamentalist vision

How I think of the fundamental domain is explained in my online lectures, Week 14, in particular the snippet Regular representation of permuting tiles. Unfortunately - if I had more time, that would have been shorter, this goes on and on, Week 15, lecture *29. Discrete symmetry factorization*, and by the time the dust settles, I do not have a gut feeling for the boundary conditions when it comes to higher-dimensional irreps (see also last week's sect. 6.1 *Discussion*).

2019-02-21 Predrag Lecture 14 Diffusion confusion

Read ChaosBook.org Chapter 24 *Deterministic diffusion*. You also might find my online lectures, Week 13 helpful. I have also added ChaosBook.org Appendix A24 *Deterministic diffusion*, but you probably do not need to read that.

7.1 Rotational random walk of a 3-spring system

Simon Berman According to the 2019 Phys. Rev. Letter of Katz-Saporta and Efrati [1], *Self-driven fractional rotational diffusion of the harmonic three-mass system*, a system of three masses connected by harmonic springs might be the simplest mechanical system (homonuclear triatomic molecule, such as ozone, except the three couplings are not the same) that exhibits a *geometric phase*. Away from its resting configuration the system is nonlinear, and once its rotational SO(2) symmetry is reduced, and as its energy is increased, it exhibits all kinds of shape-dependent chaotic geometric phases. Katz and Efrati [1] mostly do numerical simulations and plot displacement vs. time diffusion plots in its 6D phase space, like this is still early 1960's. The earlier arXiv:1706.09868 version has more information than the PRL. One suspects that a bit of thinking along periodic orbit theory lines could yield some insight into the diffusive properties of its shape-changing dynamics.

In the symmetry-reduced or the 'shape' state space there is a D_3 symmetry. One sees it in their [1] Hamiltonian (2): the b^{ij} vectors can be viewed as the three coordinates of an equilateral triangle in the w_1-w_2 plane. Since the Hamiltonian only depends on $|w|$ and in a symmetric way on $w\cdot b^{ij}$, it has a D_3 symmetry for (w_1, w_2) components of the w vector, and a reflection symmetry for w_3. So the total symmetry group is $D_3 \times C^{1/2}$.

Predrag As the system is D_3 symmetric, the symmetry should be quotiented as in (this week's lectures) and ChaosBook.org. The students from Weizmann (as well as all our local plumber apprentices) believe they have been born knowing everything, and thus they do not need to take ChaosBook.org/course1, so they would have no idea that

- they are supposed to quotient the symmetry
- probability densities (eigenfunctions of the evolution operator; Perron--Frobenius and its generalizations) block diagonalize as irreps of $D_3 = C_{3v}$, and

- that makes all calculations, numerical and periodic orbit-type more transparent and more convergent.

By going to relative w's coordinates, one has quotiented only the $2D$ Euclidean translations and $\mathrm{SO}(2)$ rotations, no discrete symmetries, so D_3 still remains. Now, anyone who has taken ChaosBook.org/course1 knows that the next step is to quotient D_3, and do the calculation in the 1/6th of the phase space, i.e., the fundamental domain.

I'm curious whether I'm right, because soon we'll look at space groups (infinite lattices with discrete symmetries) and there I have confused understanding of how to quotient the space group, but that is related to diffusion in space, rather than the angular diffusion, as in this 3-springs system.

We can make this a course project for a student in this course (a project instead of taking the final). To be especially pedagogical, we'll ask them to do it in Julia (there is one potential candidate on Piazza).

Predrag proposal: 2-body, 3-spring system We need the *simplest* illustration of a geometric phase, and its diffusion along the continuous symmetry direction induced by chaotic ("turbulent") shape-changing dynamics. So let's take one of the masses infinite. Still 3 springs, but only 2 bodies moving in a plane. We still have $\mathrm{SO}(2)$ continuous symmetry to reduce. What remains is the $\mathrm{D}_2 = \{e, \sigma\}$ symmetry of exchanging the two particles, with two irreps, the symmetric and the antisymmetric normal modes. There is shape-changing dynamics, with the potential a nonlinear function of w_j's, so for larger energies we expect angular geometric phase diffusion, but in a lower-dimensional phase space than that of the free 3-springs system. Easier to work out and look at Poincaré sections, search for relative equilibria and relative periodic orbits, compute the angular diffusion constant from its cycle expansion formulation.

References

[1] O. Katz-Saporta and E. Efrati, "Self-driven fractional rotational diffusion of the harmonic three-mass system", Phys. Rev. Lett. **122**, 024102 (2019).

Exercises

7.1. **Am I a group?** Show that multiplication table

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	b	f	c
b	b	d	e	f	c	a
c	c	b	f	e	a	d
d	d	f	c	a	e	b
f	f	c	a	d	b	e

describes a group. Or does it? (Hint: check whether this table satisfies the group axioms.)

7.2. **Product of two groups.** Let G_1 and G_2 be two finite groups. The elements of the product set $G = G_1 \times G_2$ are defined as pairs (g_1, g_2), $g_1 \in G_1$ $g_2 \in G_2$.

(a) Show that G is a group with the multiplication operation $(g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2')$.

Let D_1 be an irreducible representation of G_1 and let D_2 be an irreducible representation of G_2. For each $g = (g_1, g_2) \in G$ define $D(g) = D_1(g_1) \times D_2(g_2)$

(b) Show that $D = D_1 \times D_2$ is an irreducible representation of G. What are the characters of D?