group theory - week 3

Group representations

Georgia Tech PHYS-7143

Homework HW3 due 2019-01-29

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 3.1 *1-dimensional representation of anything*	1 point
Exercise 3.2 *2-dimensional representation of S_3*	4 points
Exercise 3.3 *3-dimensional representations of D_3*	5 points

Bonus points

Exercise 3.4 *Abelian groups*	1 point
Exercise 3.5 *Representations of C_N*	1 point

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2019-01-22 Predrag Lecture 5 Representation theory

Irreps, unitary reps and Schur's Lemma.

This lecture covers Chapter 2 *Representation Theory and Basic Theorems* of Dresselhaus *et al.* textbook [1] (click here), up to the proof of Schur's Lemma. The exposition (or the corresponding chapter in Tinkham [2]) comes from Wigner's classic *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* [3], which is a harder going, but the more group theory you learn the more you'll appreciate it. Eugene Wigner got the 1963 Nobel Prize in Physics, so by mid 60's gruppenpest was accepted in finer social circles.

2019-01-24 Predrag Lecture 6 Schur's Lemma

This lecture covers Sects. 2.5 and 2.6 *Schur's Lemma* of Dresselhaus *et al.* textbook [1] (click here).

3.1 Literature

The structure of finite groups was understood by late 19th century. A full list of finite groups was another matter. The complete proof of the classification of all finite groups takes about 3 000 pages, a collective 40-years undertaking by over 100 mathematicians, read the wiki.

From Emory Math Department: A pariah is real! The simple finite groups fit into 18 families, except for the 26 sporadic groups. 20 sporadic groups AKA the Happy Family are parts of the Monster group. The remaining six loners are known as the pariahs. (Check the previous week notes sect. 5.1 *Literature* for links to the Ree group and the whole classification.)

References

[1] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[2] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

[3] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

3.1. **1-dimensional representation of anything.** Let $D(g)$ be a representation of a group G. Show that $d(g) = \det D(g)$ is one-dimensional representation of G as well.

(B. Gutkin)

3.2. **2-dimensional representation of S_3.**

(i) Show that the group S_3 can be generated by two permutations:

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad d = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

(ii) Show that matrices:

$$\rho(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ \rho(a) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \rho(d) = \begin{pmatrix} z & 0 \\ 0 & z^2 \end{pmatrix},$$

with $z = e^{i2\pi/3}$, provide proper (faithful) representation for these elements and find representation for the remaining elements of the group.

(iii) Is this representation irreducible?

(B. Gutkin)

3.3. **3-dimensional representations of D_3.** The group D_3 is the symmetry group of the equilateral triangle. It has 6 elements

$$D_3 = \{E, C, C^2, \sigma^{(1)}, \sigma^{(2)}, \sigma^{(3)}\},$$

where C is rotation by $2\pi/3$ and $\sigma^{(i)}$ is reflection along one of the 3 symmetry axes.

(i) Prove that this group is isomorphic to S_3

(ii) Show that matrices

$$\mathcal{D}(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \mathcal{D}(C) = \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^2 \end{pmatrix}, \mathcal{D}(\sigma^{(1)}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

generate a 3-dimensional representation $\mathcal{D}$ of D_3. Hint: Calculate products for representations of group elements and compare with the group table (see lecture).

(iii) Show that this is a reducible representation which can be split into one dimensional A and two-dimensional representation Γ. In other words find a matrix R such that

$$\mathbf{R}\mathcal{D}(g)\mathbf{R}^{-1} = \begin{pmatrix} A(g) & 0 \\ 0 & \Gamma(g) \end{pmatrix}$$

for all elements g of D_3. (Might help: D_3 has only one (non-equivalent) 2-dim irreducible representation).

(B. Gutkin)

3.4. **Abelian groups.** Let G be a group with only one-dimensional irreducible representations. Show that G is Abelian.

(B. Gutkin)

3.5. **Representations of C_N.** Find all irreducible representations of C_N.

(B. Gutkin)