

## group theory - week 13

# Simple Lie algebras; $SU(3)$

**Georgia Tech PHYS-7143**

**Homework HW13**

due Tuesday 2019-04-16

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== show all your work for maximum credit,  
== put labels, title, legends on any graphs  
== acknowledge study group member, if collective effort  
== if you are LaTeXing, here is the [source code](#)

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Exercise 13.1 *Root systems of simple Lie algebras* 5 points

Exercise 13.2 *Meson octet* 5 points

**Bonus points**

Exercise 13.3  *$SU(3)$  symmetry in 3D Harmonic Oscillator* 5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

**2019-04-09 Predrag Lecture 25 Reps of simple algebras**

Gutkin notes, **Lect. 10** *Representations of simple algebras, general construction.*  
*Application to SU(3), Sects. 1-4.*

**2019-04-11 Predrag Lecture 26 Cartan's SU(3) irreps**

Gutkin notes, **Lect. 10** *Representations of simple algebras, general construction.*  
*Application to SU(3), Sect. 5.*

## **13.1 Group theory news**

**Mathematicians map  $E_8$** , and it is bigger than the human genome.

Turns out, applications of group theory go way beyond what is covered in this course:

**Group theory of defamation:** The officers argued Sawant's statements impugned them individually even though she only spoke about the police department as a whole. The court says suing as individuals and advancing a group theory of defamation takes far more than the officers showed in their complaint.

[W]hether proceeding under an individual or group theory, Plaintiffs must plead that the statements "specifically" identified or singled them out, or was understood as "referring to [them] in particular." Sims, 20 Wn. App. at 236.

EXERCISES

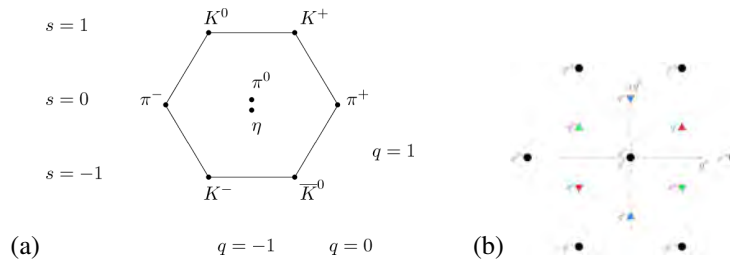


Figure 13.1: (a) The meson (pseudoscalars) octet. (b) The quark triplet, the anti-quark triplet and the gluon octet. (Wikipedia).

Exercises

13.1. Root system of simple Lie algebras.

a) Determine dimensions of Lie algebras  $\mathfrak{so}(N)$ ,  $\mathfrak{su}(N)$  and dimensions of their Cartan subalgebras. What is the number of the positive roots for these Lie algebras?

b) Show that  $N \times N$  diagonal matrices  $H_i$  with zero traces and upper/lower corner  $N \times N$  matrices  $E^{(a,b)}$  with the elements  $E_{i,j}^{(a,b)} = \delta_{ia}\delta_{jb}$  provide Cartan-Weyl basis of  $\mathfrak{su}(N)$ . To put it differently, show that  $E^{(a,b)}$  are eigenstates for adjoint representation of  $H_i$ 's.

(B. Gutkin)

13.2. Meson octet. In Gutkin lecture notes, Lect. 11 Strong interactions: flavor  $SU(3)$ , the meson octet, figure 13.1 (a)

$$\begin{aligned} \Phi &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ & 0 \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \bar{K}^0 & 0 \end{pmatrix} + \frac{\eta}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{aligned} \quad (13.1)$$

is interpreted as arising from the adjoint representation of  $SU(3)$ , i.e., the traceless part of the quark-antiquark  $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$  outer product (see figure 13.1 (b)),

$$\begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix}. \quad (13.2)$$

where we have replaced in (13.1) the constituent  $q \otimes \bar{q}$  combinations by the names of the elementary particles they build.

Given the quark quantum numbers

	$Q$	$I$	$I_3$	$Y$	$B$
u	2/3	1/2	1/2	1/3	1/3
d	-1/3	1/2	-1/2	1/3	1/3
s	-1/3	0	0	-2/3	1/3

verify the strangeness and charge assignments of figure 13.1 (a).

- 13.3. **SU(3) symmetry in 3D Harmonic Oscillator.** The Hamiltonian of 3D isotropic harmonic oscillator is given by

$$H = \sum_{i=1}^3 \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 = \hbar\omega \sum_{i=1}^3 (a_i^\dagger a_i + 1/2),$$

where  $a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + i\sqrt{\frac{1}{2m\omega\hbar}} p_i$  is creation ( $a_i^\dagger$  resp. annihilation) operator satisfying  $[a_i, a_j^\dagger] = \delta_{ij}$ ,  $[a_i, a_j] = 0$ .

- Show that  $a_i \rightarrow U_{i,j} a_j$ , with  $U \in U(3)$  is a symmetry of the Hamiltonian. In other words isotropic 3D harmonic oscillator has  $U(3)$  rather than  $O(3)$  symmetry!
- Calculate degeneracy of the  $n$ -th level  $E_n = \omega\hbar(n + 3/2)$  of the oscillator.
- By comparison of dimensions find out which representations of  $SU(3)$  appear in the spectrum of harmonic oscillator.

(B. Gutkin)