

group theory - week 11

SU(2) and SO(3)

Georgia Tech PHYS-7143

Homework HW11

due Tuesday 2019-04-02

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the [source code](#)

Exercise 11.1 <i>The characters of SO(3) representations</i>	1 point
Exercise 11.2 <i>Lie algebra of SO(4) and SU(2) \otimes SU(2)</i>	6 points
Exercise 11.5 <i>SO(n) Clebsch-Gordan series for $V \otimes V$.</i>	3 points

Bonus points

Exercise 11.3 <i>Real and pseudo-real representations of SO(3)</i>	4 points
Exercise 11.4 <i>Total spin of N particles</i>	5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2019-03-26 Predrag Lecture 21 $SU(2)$ and $SO(3)$

Gutkin notes, [Lect. 9 \$SU\(2\)\$, \$SO\(3\)\$ and their representations](#), Sects. 1-3.2; sect. [11.3 \$SU\(2\)\$ – \$SO\(3\)\$ correspondence](#) below.

2019-03-28 Predrag Lecture 22 $SO(3)$ in QM guest lecture by John Wood

Birdtrack notation [\[1\]](#) ([click here](#)) that was scheduled for this date thus never got explained. We can live without it...

To learn about it anyway, you can fetch clippings on irreps of $SU(n)$ and $SO(n)$ from Predrag's monograph [\[1\]](#) [here](#). Go through Sect. 2.2 *First example: $SU(n)$* , Sect. 6.1 *Symmetrization*, Sect. 6.2 *Antisymmetrization*, Sect. 9.1 *Two-index tensors*. Skim through Sect. 9.2 *Three-index tensors*, and Table 9.1. There is also a glimpse of a some birdtracking (still to be written up) in sect. [11.4 Irreps of \$SO\(n\)\$](#) .

Reading for week [15](#): Sect. 9.3 *Young tableaux*.

11.1 Recap of the course, so far

Read this discussion by opening it in the most recent version of P. Cvitanović, [World Wide Quest to Tame Group Theory](#) (because it is hyperlinked to earlier weeks of the course).

Predrag This course is all about *class* (physically distinct symmetry operations) and *character* (mining numbers from symmetries).

Here are some question by the dream student Henriette Roux (pseudonym) that I have answered in part in class discussions, but still have to write up:

Henriette Roux Why is it that the Fourier transformation works? The presence of a discrete but infinite translational symmetry in a system calls for its use of it to diagonalize the matrix and thus make calculations easier, but exactly *why* is the Fourier transform able to do this?

Henriette Roux How is this Fourier transform as we have studied in the space/point groups section related to that which we have derived from the projection operators?

Henriette Roux As an extension of the Fourier transform, are there any equivalent of Fourier transforms for rotations or other infinite but discrete symmetries as well? So for example, if there is a system with a discrete but infinite rotational symmetry, is there a “rotational” transform where the representing matrix is diagonalized? Are there whole classes of such transformations?

Henriette Roux You say that position and momentum are “dual” to each other, and so is the real space and reciprocal space (I guess it's the same thing as position and momentum but just for argument sake). The commonality between these are the fact that they can be Fourier transformed from one space to another. Does

this mean that unitary operations, e^{iHt} , suggest a Fourier transform from the “energy” or “frequency” space to “time” space as well?

Henriette Roux This seems very closely related to Noether’s theorem as well, is there a way to explain this similarity?

Henriette Roux The special thing about Lie groups is that there exist analytic functions which link $g(a)$ and its inverse, and $c = f(b, a)$ for $g(c) = g(a)g(b)$. Does this need for analytic functions come from the fact that to construct a group manifold, the maps relating different “local” Euclidean spaces need to be C^∞ , or smooth? If so, is there a reference we can refer to which explains how the Lie groups satisfy all the other conditions of a manifold (establishing an open ball, building an atlas and so on) as well? Just as an extension, how do you even study groups which do not fall under the realm of a manifold? Don’t common functions like differentials and integrations not apply in spaces outside a manifold?

Henriette Roux Why is that we Taylor expand the group in the first place? How is this connected to the shift to left/right group study we did?

The next few questions are about General Relativity, and how is what is covered in this course applicable to GR:

Henriette Roux We keep to the first order in the expansion for $g(\theta)$ as we are considering the tangent space to the manifold. In the context of the GR, the tangent space was defined as the space of directional derivatives at a point. In our case, we are studying groups, which are not, in general, vectors (well I guess they can be $[1 \times 1]$ vectors/matrices but that’s only specific irreps, so how do we understand the concept of tangent space as you have define it?

- Or does it work out since Lie groups are always Abelian and thus have an infinite number of 1D irreps?

- What happens if we keep the expansion to the 2nd order? Does the mathematics change in any way? Is there a good reason to ignore the 2nd and higher order expansions, not just in the physics sense (keeping to largest order of significance) but in the mathematical way of understanding things?

11.2 Central force problems

John Wood For another way of looking at the H atom (and all solvable central force problems) download my chapter ([click here](#)) from *Quantum Mechanics for Nuclear Structure: I. A Primer*, IOP science series.

The SO(2, 1) method can be extended to solve relativistic central force problems (one of my students did his Ph.D. thesis on this 20 years ago).

Q: Is the geometry associated with these algebraic structures, as applied to central force problems, explored?

11.3 SU(2) – SO(3) correspondence

Notes by Kimberly Y. Short

Angular momentum $L = r \times p$ has three components, the operators that generate SU(2) and satisfy $[L_1, L_2] = iL_3$. If we define $e = L_1 + iL_2$, $f = L_1 - iL_2$, and $h = 2L_3$, then we have the following algebra:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h \quad (11.1)$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (11.2)$$

Matrices e and f act as the raising and lowering (also called ‘ladder’) operators L_{\pm} in this representation. (The set $\{e, f, h\}$ forms an ‘ sl_2 -triple’.)

We observe that there are $n^2 - 1 = 3$ such operators satisfying this algebra, which is the Lie algebra of SU(n), where $n = 2$. The eigenvalues of h are integers separated by 2, and the eigenvalues of L_3 must be half-integers separated by 1. Consequently, the representation with highest L_3 eigenvalue given by l must have dimension $2l + 1$ (note: $2l$ is λ_{max} for h).

Further, $L^2 = L \cdot L$ commutes with L_1 , L_2 , and L_3 and hence, by Schur’s Lemma, $L^2 = \lambda \mathbb{I}$ in this representation, so every vector is an eigenvector of L^2 . For example, we’ve seen in quantum mechanics,

$$L^2 Y_l^m = l(l+1)\hbar^2 Y_l^m \quad (11.3)$$

And since the spherical harmonics $Y_l^m(\theta, \phi)$ constitute an orthonormal basis of the Hilbert space of square-integrable functions, any vector can be expanded in a basis of $Y_l^m(\theta, \phi)$. L_{\pm} act on Y_l^m in the following way:

$$L_{\pm} Y_l^m = \hbar \sqrt{l(l+1) - m(m \pm 1)} Y_l^{m \pm 1}. \quad (11.4)$$

An element of SU(2) can be written as

$$e^{i\sigma_j \alpha_j / 2} \quad (11.5)$$

where σ_j is a Pauli matrix and α_j is a number. (The exponentiation of the Pauli matrices gives SU(2).) What is the importance of the 1/2 factor in the argument of the exponential. First, consider a generic position vector $\mathbf{r} = x\hat{e}_i + y\hat{e}_j + z\hat{e}_k$. We may construct a matrix of the form

$$\begin{aligned} \sigma \cdot r &= \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix} + \begin{pmatrix} 0 & -iy \\ iy & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & -z \end{pmatrix} \\ &= \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \end{aligned} \quad (11.6)$$

The determinant,

$$\det \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} = -(x^2 + y^2 + z^2) = -\mathbf{x}^2 \quad (11.7)$$

is an expression for the length of a vector.

Now consider a unitary transformation of this matrix. For example,

$$\begin{aligned} U(\sigma \cdot \mathbf{r})U^\dagger &= \sigma_x(\sigma \cdot \mathbf{r})\sigma_x \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z & x - iy \\ x - iy & -z \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} \end{aligned} \quad (11.8)$$

Taking this determinant, we find the same expression as before:

$$\det \begin{pmatrix} -z & x + iy \\ x - iy & z \end{pmatrix} = -(x^2 + y^2 + z^2) = -\mathbf{x}^2 \quad (11.9)$$

We observe that, like SO(3), SU(2) preserves the lengths of vectors.

The correspondence between SO(3) and SU(2) can be made more explicit. To see this, consider an SU(2) transformation on a two-component object called a *spinor* ψ where

$$\psi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad (11.10)$$

and

$$x = \frac{1}{2}(\beta^2 - \alpha^2), \quad y = -\frac{i}{2}(\alpha^2 + \beta^2), \quad z = \alpha\beta. \quad (11.11)$$

One may check that an SU(2) transformation on ψ is equivalent to an SO(3) transformation on \mathbf{x} . From this equivalence, one sees that an SU(2) transformation has three real parameters that correspond to the three rotation angles of SO(3). If we label the “angles” for the SU(2) transformation by α , β , and γ , we observe, for a “rotation” about \hat{x}

$$U_x(\alpha) = \begin{pmatrix} \cos \alpha/2 & i \sin \alpha/2 \\ i \sin \alpha/2 & \cos \alpha/2 \end{pmatrix}. \quad (11.12)$$

Likewise for an SU(2) transformation about \hat{y} :

$$U_y(\beta) = \begin{pmatrix} \cos \beta/2 & \sin \beta/2 \\ -\sin \beta/2 & \cos \beta/2 \end{pmatrix} \quad (11.13)$$

And for the final rotation, the SU(2) transformation about \hat{z} :

$$U_z(\gamma) = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \quad (11.14)$$

Compare these three matrices to the corresponding SO(3) rotation matrices:

$$R_x(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \zeta & \sin \zeta \\ 0 & -\sin \zeta & \cos \zeta \end{pmatrix}, \quad R_y(\phi) = \begin{pmatrix} \cos \phi & 0 & \sin \phi \\ 0 & 1 & 0 \\ -\sin \phi & 0 & \cos \phi \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (11.15)$$

They're equivalent! Result: *Half the rotation angle generated by SU(2) corresponds to a rotation generated by SO(3).*

In this context, the eigenvalue equation for L_3 and for L^2 are differential equations whose solutions are the spherical harmonics Y_l^m which take the form

$$e^{im\phi} P_l^m(\cos \theta), \quad -l \leq m \leq l \quad (11.16)$$

in spherical coordinates and which determine the shape of electron orbitals and their probabilities to be found in a given region.

In quantum mechanics, the possible results of a measurement are determined by the possible eigenvalues of an operator. As such, the possible measurable values of the z -component of angular momentum correspond to the allowed values of L_3 . The measurement outcomes are not arbitrary; the largest one, l , must be a half-integer, and there are $2l + 1$ eigenvectors. Applying the lowering operator L_- one-by-one, we can find the possible outcomes to be $m \in \{l, l - 1, \dots, -l\}$. The angular dependence of the corresponding wave function goes as $\sim e^{im\phi} P_l^m(\cos \theta)$. In addition, higher values of l correspond to higher energy, so the different values of l correspond to different electron orbitals in order of increasing energy.

11.4 Irreps of SO(n)

The dimension of SO(n) is given by the trace of the adjoint projection operator:

$$N = \text{tr } \mathbf{P}_A = \text{tr} \begin{pmatrix} \circ & & \\ & \bullet & \\ & & \circ \end{pmatrix} = \frac{n(n-1)}{2}. \quad (11.17)$$

Dimensions of the other reps are listed in table 11.1.

References

- [1] P. Cvitanović, *Group Theory: Birdtracks, Lie's and Exceptional Groups* (Princeton Univ. Press, Princeton NJ, 2004).

EXERCISES

Young tableaux	$\square \times \square =$	\bullet	+	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	+	$\begin{array}{ c c } \hline \square & \square \\ \hline \end{array}$
Dimensions	$n^2 =$	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Projectors	$\frac{1}{n} \begin{array}{ c } \hline \square \\ \hline \end{array}$	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	+	$\frac{1}{n} \begin{array}{ c } \hline \square \\ \hline \end{array}$	+	$\left\{ \begin{array}{ c c } \hline \square & \square \\ \hline \end{array} - \frac{1}{n} \begin{array}{ c } \hline \square \\ \hline \end{array} \right\} \begin{array}{ c } \hline \square \\ \hline \end{array}$

Table 11.1: $SO(n)$ Clebsch-Gordan series for $V \otimes V$.

Exercises

- 11.1. **The characters of $SO(3)$ representations:** Show that for an irrep labeled by j , the character of a conjugacy class labeled by θ

$$\chi^{(j)}(\theta) = \frac{\sin(j + 1/2)\theta}{\sin(\theta/2)} \tag{11.18}$$

can be obtained by taking the trace of $R_z^j(\theta)$. Verify that for $j = 1$ this character is the three dimensional special orthogonal representation character (10.6).

- 11.2. **Lie algebra of $SO(4)$ and $SU(2) \otimes SU(2)$.** One particle Hamiltonian with a central potential has in general $SO(3)$ symmetry group. It turns out, however, that for Coulomb potential the symmetry group is actually larger - $SO(4)$, rather than $SO(3)$. This explains why the energy level degeneracies in the hydrogen atom are anomalously large. So $SO(4)$ and its representations are of a special importance in atomic physics.

- (a) Show that the Lie algebra $\mathfrak{so}(4)$ of the group $SO(4)$ is generated by real antisymmetric 4×4 matrices.
- (b) What is the dimension of $\mathfrak{so}(4)$?

A natural basis in $\mathfrak{so}(4)$ is provided by antisymmetric matrices $M_{\mu\nu}$, $\mu, \nu \in 1, 2, 3, 4$, $\mu \neq \nu$, generators of $SO(4)$ rotations which leave invariant the $\mu\nu$ -plane. The elements of these matrices are given by

$$(M_{\mu\nu})_{ij} = \delta_{i\mu}\delta_{j\nu} - \delta_{j\mu}\delta_{i\nu} \tag{11.19}$$

- (c) Check that these matrices satisfy the commutation relationship

$$[M_{ab}, M_{cd}] = M_{ad}\delta_{bc} + M_{bc}\delta_{ad} - M_{ac}\delta_{bd} - M_{bd}\delta_{ac}. \tag{11.20}$$

- (d) Show that Lie algebras of the groups $SO(4)$ and $SU(2) \times SU(2)$ are isomorphic.
Path:

- (d.i) Define matrices

$$J_k = \frac{1}{2} \varepsilon_{kij} M_{i,j}, \quad K_k = M_{k4}, \quad k = 1, 2, 3$$

and

$$A_k = \frac{1}{2} (J_k + K_k) \quad \text{and} \quad B_k = \frac{1}{2} (J_k - K_k).$$

- (d.ii) Show that \mathcal{A} and \mathcal{B} satisfy the same commutation relations as two copies of $\mathfrak{su}(2)$.
- (e) How does one construct irreps of $\mathfrak{so}(4)$ out of irreps of $\mathfrak{su}(2)$?
- (f) Are groups $SO(4)$ and $SU(2) \otimes SU(2)$ isomorphic to each other?

(B. Gutkin)

11.3. **Real and pseudo-real representations of $SO(3)$.** Recall (Gutkin notes, [Lect. 4 Representation Theory II](#), Sect. 5.5. *Three types of representations*) that there exist three types of representation which can be distinguished by the indicator:

$$\int_G d\mu(g) \chi_l(g^2) = \begin{cases} +1 & \text{real} \\ 0 & \text{complex} \\ -1 & \text{pseudo-real} \end{cases} . \quad (11.21)$$

Determine for which values of $l = 0, 1/2, 1, 3/2, 2, \dots$ the representation D_l of $SO(3)$ is real or pseudo-real.

Hint: The characters and Haar measure (10.8) of $SO(3)$ are given by

$$\chi_l(g) = \frac{\sin\left(\left[l + \frac{1}{2}\right]\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, \quad d\mu(g) = \frac{d\theta}{\pi} \sin^2(\theta/2) \quad (11.22)$$

where θ is rotation angle for the group element g .

(B. Gutkin)

11.4. **Total spin of N particles.** Consider a system of four particles with spin $1/2$. Assuming that all (except spin) degrees of freedom are frozen the Hilbert space of the system is given by $V = V_{1/2} \otimes V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$, with $V_{1/2}$ being two-dimensional space for each spin. $V = \oplus V_s$ can be decomposed then into different sectors V_s having the total spin s i.e., $\hat{S}^2 v = s(s+1)v$, for any $v \in V_s$. Here $\hat{S}^2 = (\sum_{i=1}^4 \hat{s}_i)^2$ and $\hat{s}_i = (\hat{s}_i^x, \hat{s}_i^y, \hat{s}_i^z)$ is spin operator for i -th particle.

- (a) What are possible values s for the total spin of the system?
- (b) Determine dimension of the subspace of V_0 with 0 total spin. In other words: how many times trivial representation enters into product:

$$D = D_{1/2} \otimes D_{1/2} \otimes D_{1/2} \otimes D_{1/2} ? \quad (11.23)$$

- (c) What is the answer to the above questions for N spins?

Hint: it is convenient to use (11.22) to decompose D into irreps.

(B. Gutkin)

11.5. **$SO(n)$ Clebsch-Gordan series for $V \otimes V$.**

- (a) Show that the product of two n -dimensional reps of $SO(n)$ decomposes into three irreps:

$$\text{---} = \frac{1}{n} \text{---} \cup + \text{---} \text{---} + \left\{ \text{---} \text{---} - \frac{1}{n} \text{---} \cup \text{---} \right\} . \quad (11.24)$$

- (b) Compute the dimensions of the three irreps.
- (c) Which one is the adjoint one, and why? Hint: check the invariance condition.

EXERCISES

11.6. **Splitting of degeneracies in a central potential.** Hamiltonian H_0 has rotational symmetry of $SO(3)$.

(a) What are the possible energy level degeneracies of H_0 ?

A weak perturbation V with a symmetry T_d of full tetrahedron group is added (e.g., V is a potential created by lattice of atoms with a symmetry of T_d).

(b) What will be the degeneracies of new Hamiltonian $H_0 + V$?

(c) Assuming that the total angular momentum of the system before the perturbation is $l = 2$. How the degeneracies of the corresponding energy level will be split after the perturbation is applied?

(B. Gutkin)

11.7. **Quadrupole transitions.**

a) Write $Q_1 = xy$, $Q_2 = zy$, $Q_3 = x^2 - y^2$ and $Q_4 = 2z^2 - x^2 - y^2$ as components of spherical tensor of rank 2. *Hint:* use spherical harmonics $Y_l^m(\theta, \varphi)$.

b) The last quantity Q_4 is known as quadrupole moment. What are the selection rules for transitions induced by Q_4 in a system with $SO(3)$ symmetry? In other words, for which m, l and k, j the transition rates:

$$P_{m,l \rightarrow k,j} \sim |\langle m l | Q_4 | j k \rangle|^2$$

are non-zero?

c) By using Wigner-Eckart theorem write down the relationship between $|\langle m l | Q_4 | j k \rangle|^2$ and $|\langle m l | Q_1 | j k \rangle|^2$ in terms of Clebsch-Gordan coefficients.

(B. Gutkin)