group theory - week 4

Hard work builds character

Georgia Tech PHYS-7143

Homework HW4 due Tuesday, September 19, 2017

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 4.3 *All irreducible representations of* D_4 10 points

Bonus points
Exercise 4.4 *Irreducible representations of dihedral group* D_n 2 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2017-09-12 Tropic Depression Irma Lecture 7

Character orthogonality theorem

Please study by yourself the Dresselhaus [1] (click here) sects. 2.7 "Wonderful Orthogonality Theorem," 2.8 "Representations and vector spaces," 3.1 "Definition of Character" and 3.2 "Characters and Class." Tinkham [4] covers the same material in Chapter 3 *Theory of Group Representations* in a more compact way.

If you have any questions you would like discussed in the class, please email them to Predrag by Thursday morning.

2017-09-14 Predrag Lecture 8

Hard work builds character

Complete Dresselhaus *et al.* [1] (click here) sects. 3.3 "Wonderful Orthogonality Theorem for Characters" to 3.8 "Setting up Character Tables". This material is also covered in Tinkham [4] Chapter 3 *Theory of Group Representations*.

4.1 Literature

I enjoyed reading Mathews and Walker [3] Chap. 16 *Introduction to groups*. You can download it from here. Goldbart writes that the book is "based on lectures by Richard Feynman at Cornell University." Very clever. Try working through the example of fig. 16.2: it is very cute, you get explicit eigenmodes from group theory alone. The main message is that if you think things through first, you never have to go through using explicit form of representation matrices - thinking in terms of invariants, like characters, will get you there much faster.

You might find Gutkin notes useful:

Lect. 4 *Representation Theory II*, up to Sect. 4.5 *Three types of representations*: Character tables. Dual character orthogonality. Regular Representation. Indicators for real, pseudo-real and complex representations. See example 4.1 "Irreps for quaternion multiplication table."

Lect. 5 *Applications I. Vibration modes* go through Wigner's theorem, C_n symmetry and D_3 symmetry. Study Example 5.1. C_n symmetry. More quantum mechanics applications follow in

Lect. 6 *Applications II. Quantum Mechanics*, Sect. *2. Perturbation theory*.

Does the proof in the Lect. 4 *Representation Theory II Appendix* that the number of irreps equals the number of classes make sense to you? For an easy argument, see Vedensky Theorem 5.2 *The number of irreducible representations of a group is equal to the number of conjugacy classes of that group.* For a proof, work though Murnaghan Theorem 7. If you prefer a proof that your professor cannot understand, click here.

Example 4.1. ***Quaternions:*** *Quaternion multiplication table is*

$$\{\pm 1, \pm i, \pm j, \pm k\} \qquad i^2 = j^2 = k^2; \quad ij = k.$$

This group has five conjugate classes:

$$\{1\}, \{-1\}, \{\pm i\}, \{\pm j\}, \{\pm k\}.$$

The only possible solution for the equation $\sum_{i=1}^{5} m_i^2 = 8$ *is* $m_i = 1, i = 1, \dots 4$*,* $m_5 = 2$*. In addition to fully symmetric representation, the other three one-dimensional representations are easy to find:* $\chi(1) = 1, \chi(-1) = 1$*, while* $\chi(i) = -1, \chi(j) = -1, \chi(k) = 1$*;* $\chi(i) = -1, \chi(k) = -1, \chi(j) = 1$ *or* $\chi(k) = -1, \chi(j) = -1, \chi(i) = 1$*. The two-dimensional representation can be find by the orthogonality relation:*

$$2 + \chi(-1) \pm \chi(k) \pm \chi(i) \pm \chi(j) = 0, \Longrightarrow \chi(-1) = -2, \chi(k) = \chi(i) = \chi(j) = 0\,.$$

Since the indicator equals

$$Ind = (2\chi(1) + 6\chi(-1))/8 = -1,$$

the last representation is pseudo-real. Note that this representation can be realized using Pauli matrices:

$$\{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}.$$

References

[1] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[2] L. Landau and E. Lifshitz, *Quantum Mechanics: Non-Relativistic Theory* (Pergamon Press, Oxford, 1959).

[3] J. Mathews and R. L. Walker, *Mathematical Methods of Physics* (W. A. Benjamin, Reading, MA, 1970).

[4] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

Exercises

4.1. **Characters of D_3.** (continued from exercise 2.4) $D_3 \cong C_{3v}$, the group of symmetries of an equilateral triangle: has three irreducible representations, two one-dimensional and the other one of multiplicity 2.

(a) All finite discrete groups are isomorphic to a permutation group or one of its subgroups, and elements of the permutation group can be expressed as cycles. Express the elements of the group D_3 as cycles. For example, one of the rotations is (123), meaning that vertex 1 maps to 2, $2 \to 3$, and $3 \to 1$.

(b) Use your representation from exercise 2.4 to compute the D_3 character table.

(c) Use a more elegant method from the group-theory literature to verify your D_3 character table.

(d) Two D_3 irreducible representations are one dimensional and the third one of multiplicity 2 is formed by $[2\times 2]$ matrices. Find the matrices for all six group elements in this representation.

4.2. **Decompose a representation of S_3.** Consider a reducible representation $D(g)$, i.e., a representation of group element g that after a suitable similarity transformation takes form

$$D(g) = \begin{pmatrix} D^{(a)}(g) & 0 & 0 & 0 \\ 0 & D^{(b)}(g) & 0 & 0 \\ 0 & 0 & D^{(c)}(g) & 0 \\ 0 & 0 & 0 & \ddots \end{pmatrix} ,$$

with character for class $\mathcal{C}$ given by

$$\chi(\mathcal{C}) = c_a\,\chi^{(a)}(\mathcal{C}) + c_b\,\chi^{(b)}(\mathcal{C}) + c_c\,\chi^{(c)}(\mathcal{C}) + \cdots ,$$

where c_a, the multiplicity of the ath irreducible representation (colloquially called "irrep"), is determined by the character orthonormality relations,

$$c_a = \overline{\chi^{(a)*}\,\chi} = \frac{1}{h}\sum_k^{class} N_k \chi^{(a)}(\mathcal{C}_k^{-1})\,\chi(\mathcal{C}_k)\,. \qquad (4.1)$$

Knowing characters is all that is needed to figure out what any reducible representation decomposes into!

As an example, let's work out the reduction of the matrix representation of S_3 permutations. The identity element acting on three objects $[a\;b\;c]$ is a 3×3 identity matrix,

$$D(E) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Transposing the first and second object yields $[b\;a\;c]$, represented by the matrix

$$D(A) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

since

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}$$

1. Find all six matrices for this representation.
2. Split this representation into its conjugacy classes.
3. Evaluate the characters $\chi(\mathcal{C}_j)$ for this representation.
4. Determine multiplicities c_a of irreps contained in this representation.
5. (bonus) Construct explicitly all irreps.
6. (bonus) Explain whether any irreps are missing in this decomposition, and why.

4.3. **All irreducible representations of D_4.** Dihedral group D_4, the symmetry group of a square, consists of 8 elements: identity, rotations by $\pi/2$, π, $3\pi/2$, and 4 reflections across symmetry axes: $D_4 = \langle g, \sigma | g^4 = \sigma^2 = e, g\sigma = \sigma g^3\rangle$

(a) Find all conjugacy classes.

(b) Determine the dimensions of irreducible representations using the relationship

$$\sum_i d_i^2 = |G|, \tag{4.2}$$

where d_i is the dimension of ith irreducible representation.

(c) Determine the remaining items of the character table.

(d) Compare with the character table of quaternions, example 4.1. Are they the same or different?

(e) Determine the indicators for all irreps of D_4. Are they the same as for the irreps of the quaternion group?

If you are at loss how to proceed, take a look at Landau and Lifschitz [2] *Vol.3: Quantum Mechanics*

(Boris Gutkin)

4.4. **Irreducible representations of dihedral group D_n.**

(a) Determine the dimensions of all irreps of dihedral group D_n, n odd.

(b) Determine the dimensions of all irreps of dihedral group D_n, n even.

This exercise is meant to be easy - guess the answer from the irreps dimension sum rule (4.2), and what you already know about D_1, D_3 and D_4. Working out also D_2 case (cut a disk into two equal halves) might be helpful. A more serious attempt would require counting conjugacy classes first. This exercise might help you later, when you are looking at irreps of the orthogonal groups $O(n)$; turns out they are different for n odd or even n, and that has physical consequences: what you learn by working out a problem in 2 dimensions might be misleading for working it out in 3 dimensions.

4.5. **Perturbation of T_d symmetry.**

A non-relativistic charged particle moves in an infinite bound potential $V(x)$ with T_d symmetry. Consult exercise 5.1 *Vibration Modes of CH_4* for the character table and other T_d details.

(a) What are the degeneracies of the quantum energy levels? How often do they appear relative to each other (i.e., what is the level density)?

A weak constant electric field is now added now along one of the $2\pi/3$ rotation axes, splitting energy levels into multiplets.

(b) What is the symmetry group of the system now?

(c) How are the levels of the original system split? What are the new degeneracies?

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4.6. **Two particles in a potential.**

Two distinguishable particles of the same mass move in a 2-dimensional potential $V(r)$ having D_4 symmetry. In addition they interact with each other with the term $\lambda W(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)$.

(a) What is the symmetry group of the Hamiltonian if $\lambda = 0$? If $\lambda \neq 0$?.

(b) What are the degeneracies of the energy levels if $\lambda = 0$?

(c) Assuming that $\lambda \ll 1$ (weak interaction), describe the energy level structure, i.e., degeneracies and quasi-degeneracies of the energy levels. What will be the answer if the interaction is strong?

Hint: when interaction is weak we can think about it as perturbation. (Boris Gutkin)