group theory - week 2

Finite groups - definitions

Georgia Tech PHYS-7143

Homework HW2 due Tuesday, September 5, 2017

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 2.1 $G_x \subset G$	1 point
Exercise 2.2 *Transitivity of conjugation*	1 point
Exercise 2.3 *Isotropy subgroup of* gx	1 points
Exercise 2.5 C_4*-invariant potential*	7 (+2) points

Total of 10 points = 100 % score.

Bonus points
Exercise 2.X: fix the errors in example 2.3 *Vibrational spectra of molecules*.

LaTeX source code	3 points
Exercise 2.8 *Three masses on a loop*	6 points
Exercise 2.7 *An arrangement of five particles*	4 points

Extra points accumulate, can help you later if you miss a few problems.

2017-08-29 Predrag Lecture 3 Don't wonna know group theory

Today's example 2.3 whiteboard derivation of normal-modes of the ring of N asymmetric pairs of oscillators is taken from Gutkin lecture notes example 5.1 C_n *symmetry*. The corresponding projection operators (1.31) are worked out in example 2.4.

2017-08-31 Predrag Lecture 4 Finite groups

Groups, permutations, rearrangement theorem, subgroups, cosets, all exemplified by the $S_3 = C_{3v} = D_3$ symmetries of an equilateral triangle. This lecture follows closely Chapter 1 *Basic Mathematical Background: Introduction* of Dresselhaus *et al.* textbook [1] (click here, ask for password if you have forgotten it). This book (or Tinkham [3]) is good on discrete and space groups, but perhaps not so good on continuous groups. The MIT course 6.734 online version contains much of the same material.

If instead, bedside crocheting is your thing, click here.

2.1 Using group theory without knowing any

It's a matter of no small pride for a card-carrying dirt physics theorist to claim full and total ignorance of group theory (read sect. A.6 *Gruppenpest* of ref. [2]). So what we will do first is work out a few examples of physical applications of group theory that you already know without knowing that you have been using "Group Theory."

Example 2.1. ***Discrete symmetries in physics:***

- *Point groups i.e., subgroups of $O(3)$.*
- *Point groups + discrete translations e.g., symmetry groups of crystals.*
- *Permutation groups*

$$S\Psi(x_1, x_2, \dots x_n) = \Psi(x_2, x_1, \dots x_n).$$

- *Boson wave functions are symmetric while fermion wave functions are anti-symmetric under exchange of variables.*

(B. Gutkin)

Example 2.2. ***Reflection and discrete rotation symmetries:***

(a) Reflection symmetry $V(x) = PV(x) = V(-x)$:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E_n\psi(x) \tag{2.1}$$

(see figure 2.1). If $\psi(x)$ is solution then $P\psi(x)$ is also solution. From this and non-degeneracy of the spectrum follows that either $P\psi(x) = \psi(x)$ or $P\psi(x) = -\psi(x)$. The first case corresponds to symmetric functions while the second one to anti-symmetric one. Thus the whole spectrum can be decomposed in accordance to a symmetry of the Hamiltonian (equations of motion).

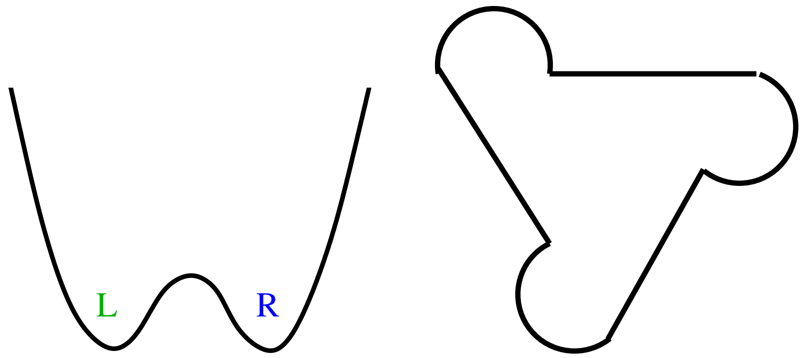


Figure 2.1: (left) A reflection-symmetric double-well potential. (right) A $1/3$rd-circle rotation-symmetric plane billiard (infinite wall potential in $2D$). (B. Gutkin)

(b) Rotation symmetry $V(x) = gV(x)$, $G = \{e, g, g^2\}$: By the same argument we have three possibilities:

$$g\psi(x) = \psi(x); \qquad g\psi(x) = e^{i2\pi/3}\psi(x); \qquad g^{-1}\psi(x) = e^{-i2\pi/3}\psi(x).$$

In addition, by the time reversal symmetry if $\psi(x)$ is solution then $\psi^*(x)$ is solution with the same eigenvalue as well. From this follows that the spectrum must be degenerate. The spectrum is split into a real eigenfunction $\{\psi_1(x)\}$, and a degenerate pair of real eigenfunctions

$$\psi_2(x) = \psi(x) + \psi^*(x); \psi_3(x) = i(\psi(x) - \psi^*(x)) , \text{ where } g\psi(x) = e^{i2\pi/3}\psi(x)$$

invariant under rotations by 1/3-rd of a circle.

(B. Gutkin)

Example 2.3. ***Vibrational spectra of molecules:*** *In the linear, harmonic oscillator approximation the classical dynamics of the molecule is governed by the Hamiltonian*

$$H = \sum_{i=1}^{N} \frac{m_i}{2}\dot{x}_i^2 + \frac{1}{2}\sum_{i,j=1}^{N} x_i^\top V_{ij} x_j ,$$

where $\{x_i\}$ are small deviations from the resting the equilibrium, resting points of the molecules labelled i. V_{ij} is a symmetric matrix, so it can be brought to a diagonal form by an orthogonal transformation, a set of N uncoupled harmonic oscillators or normal modes *of frequencies $\{\omega_i\}$.*

$$x \to y = Ux, \qquad H = \sum_{i=1}^{N} \frac{m_i}{2}\left(\dot{y}_i^2 + \omega_i^2 y_i^2\right) . \tag{2.2}$$

Consider now the ring of pair-wise interactions of two kinds of molecules sketched in figure 2.2 (a), given by the potential

$$V(z) = \frac{1}{2}\sum_{i=1}^{N}\left(k_1(x_i - y_i)^2 + k_2(x_{i+1} - y_i)^2\right) , \qquad z_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} , \tag{2.3}$$

whose $[2N\times 2N]$ matrix form is (aside for the cognoscenti: kind of a Toeplitz matrix):

$$V_{ij}=\frac{1}{2}\begin{pmatrix} k_1+k_2 & -k_1 & 0 & 0 & 0 & \dots & 0 & 0 & -k_2 \\ -k_1 & k_1+k_2 & -k_2 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & -k_2 & k_1+k_2 & -k_1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & -k_1 & k_1+k_2 & -k_2 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & -k_2 & k_1+k_2 & -k_1 \\ -k_2 & 0 & 0 & 0 & 0 & \dots & 0 & -k_1 & k_1+k_2 \end{pmatrix}$$

This potential matrix is a holy mess. How do we find an orthogonal transformation (2.2) that diagonalizes it? Look at figure 2.2 (a). Molecules lie on a circle, so that suggests we should use a Fourier representation. As the $i=1$ labelling of the starting molecule on a ring is arbitrary, we are free to relabel them, for example use the next molecule pair as the starting one. This relabelling is accomplished by the $[2N\times 2N]$ permutation matrix (or 'one-step shift', 'stepping' or 'translation' matrix) M of form

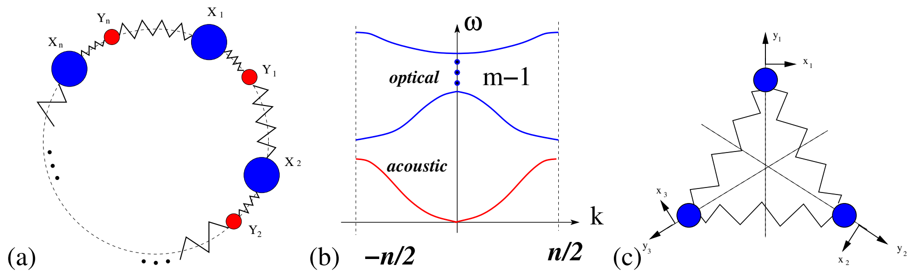


Figure 2.2: (a) Chain with circular symmetry. (b) Dependance of frequency on the representation wavenumber k. (c) Molecule with D_3 symmetry. (B. Gutkin)

$$\underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & I \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{pmatrix}}_{M}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ \vdots \\ z_n \end{pmatrix}=\begin{pmatrix} z_n \\ z_1 \\ z_2 \\ \vdots \\ z_{n-1} \end{pmatrix}, \qquad I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad z_i=\begin{pmatrix} x_i \\ y_i \end{pmatrix} \tag{2.4}$$

Projection operators corresponding to M are worked out in example 2.4. They are N distinct $[2N\times 2N]$ matrices,

$$P_k=\begin{pmatrix} I & \bar\lambda I & \bar\lambda^2 I & \dots & \bar\lambda^{N-2} I & \bar\lambda^{N-1} I \\ \lambda I & I & \bar\lambda I & \dots & \bar\lambda^{N-3} I & \bar\lambda^{N-2} I \\ \lambda^2 I & \lambda I & I & \dots & \bar\lambda^{N-4} I & \bar\lambda^{N-3} I \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda^{N-2} I & \lambda^{N-3} I & \lambda^{N-4} I & \dots & I & \bar\lambda I \\ \lambda^{N-1} I & \lambda^{N-2} I & \lambda^{N-2} I & \dots & \lambda I & I \end{pmatrix}, \qquad \lambda=\exp\left(\frac{2\pi i}{N}k\right) \tag{2.5}$$

which decompose the $2N$-dimensional configuration space of the molecule ring into a direct sum of N 2-dimensional spaces, one for each discrete Fourier mode $k = 0, 1, 2, \cdots, N-1$.

The system (2.3) is clearly invariant under the cyclic permutation relabelling M, $[V, M] = 0$ (though checking this by explicit matrix multiplications might be a bit tedious), so the P_k decompose the the interaction potential V as well, and reduce its action to the kth 2-dimensional subspace. Thus the $[2N \times 2N]$ diagonalization (2.2) is now reduced to a $[2 \times 2]$ diagonalization which one can do by hand. The resulting kth space is spanned by two $2N$-dimensional vectors, which we guess to be of form:

$$\eta_1 = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 0 \\ \lambda \\ 0 \\ \vdots \\ \lambda^{n-1} \\ 0 \end{pmatrix}, \qquad \eta_2 = \frac{1}{\sqrt{n}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ \lambda \\ \vdots \\ 0 \\ \lambda^{n-1} \end{pmatrix}.$$

In order to find eigenfrequences we have to consider action of V on these two vectors:

$$V\eta_1 = (k_1 + k_2)\eta_1 - (k_1 + k_2\lambda)\eta_2 \,, \qquad V\eta_2 = (k_1 + k_2)\eta_2 - (k_1 + k_2\bar{\lambda})\eta_1 \,.$$

The corresponding eigenfrequencies are determined by the equation:

$$\begin{aligned} 0 &= \det\left(\begin{pmatrix} k_1 + k_2 & -(k_1 + k_2\lambda) \\ -(k_1 + k_2\bar{\lambda}) & k_1 + k_2 \end{pmatrix} - \frac{\omega^2}{2} I \right) \qquad \Longrightarrow \\ \frac{1}{2}\omega_\pm^2(k) &= k_1 + k_2 \pm |k_1 + k_2\lambda^k| \,, \end{aligned} \tag{2.6}$$

one acoustic ($\omega(0) = 0$), one optical, see figure 2.2 (b) and the acoustic and optical phonons wiki. *(B. Gutkin)*

Example 2.4. ***Projection operators for cyclic group C_N.***

Consider a cyclic group $C_N = \{e, g, g^2, \cdots g^{N-1}\}$, and let $M = D(g)$ be a $[2N \times 2N]$ representation of the one-step shift g. In the projection operator formulation (1.31), the N distinct eigenvalues of M, the Nth roots of unity $\lambda_n = \lambda^n$, $\lambda = \exp(i\, 2\pi/N)$, $n = 0, \ldots N-1$, split the $2N$-dimensional space into N 2-dimensional subspaces by means of projection operators

$$P_n = \prod_{m \neq n} \frac{M - \lambda_m I}{\lambda_n - \lambda_m} = \prod_{m=1}^{N-1} \frac{\lambda^{-n} M - \lambda^m I}{1 - \lambda^m} \,, \tag{2.7}$$

where we have multiplied all denominators and numerators by λ^{-n}. The numerator is now a matrix polynomial of form $(x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$, with the zeroth root $(x - \lambda^0) = (x-1)$ quotiented out from the defining matrix equation $M^N - 1 = 0$. Using

$$\frac{1 - x^N}{1 - x} = 1 + x + \cdots + x^{N-1} = (x-\lambda)(x-\lambda^2)\cdots(x-\lambda^{N-1})$$

we obtain the projection operator in form of a discrete Fourier sum (rather than the product (1.31)),

$$P_n = \frac{1}{N} \sum_{m=0}^{N-1} e^{i\,\frac{2\pi}{N} nm} M^m \,.$$

This form of the projection operator is the simplest example of the key group theory tool, projection operator

$$P_n = \frac{1}{|G|}\sum_{g\in G}\bar{\chi}(g)D(g)$$

upon which stands all that follows in this course. *(B. Gutkin and P. Cvitanović)*

Example 2.5. D_3 ***symmetry:*** *Reflactions and rotations of a triangle, figure 2.2 (c)*

$$D(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \qquad D(\sigma_1) = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \tag{2.8}$$

$$D(\sigma_2) = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad D(\sigma_3) = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.9}$$

$$G = \{[e];[g,g^2];[\sigma_1,\sigma_2,\sigma_3]\}, \qquad \chi^{(1)} = \{1,1,1\}, \chi^{(2)} = \{1,1,-1\}, \chi^{(3)} = \{2,-1,0\}$$

$$r_i = \chi(e)\chi^{(i)}(e)/6; \qquad r_i = \{1,1,2\} \quad\Longrightarrow\quad D = 2E \oplus A_1 \oplus A_2.$$

$$P_i = \frac{1}{3}\sum_{g\in G}\chi^{(i)}(g)D(g)$$

$$P_1 = \frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad P_2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.10}$$

The vibrational modes associated with the two 1-dimensional representations are given by

$$P_1 V = \alpha \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{and} \qquad P_2 V = \beta \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

respectively. Here $P_1 V$ represents symmetric mode shown in figure 2.3 (red). The second mode $P_2 V$ corresponds to the rotations of the whole system. Finally the projection operator for the two-dimensional representation is

$$P_3 = \frac{2}{6}(2D(I) - D(T) - D(T^2)) = \frac{1}{3}\begin{pmatrix} 2 & 0 & -1 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 & 0 & -1 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 2 & 0 & -1 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & -1 & 0 & 2 \end{pmatrix} \tag{2.11}$$

From this we have to separate two vectors corresponding to shift in x and y directions.

$$\eta_x = \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \end{pmatrix}, \qquad \eta_y = \begin{pmatrix} 0 \\ 1 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

$$P_3 V = \left\{ \underbrace{\alpha \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{\xi_1} + \underbrace{\beta \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}}_{\xi_2} + \underbrace{\gamma \frac{1}{\sqrt{6}} \begin{pmatrix} 0 \\ 2 \\ 0 \\ -1 \\ 0 \\ -1 \end{pmatrix}}_{\xi_3} + \underbrace{\delta \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}}_{\xi_4} \right\},$$

where $\eta_x = \sqrt{3/2}(\xi_4 + \xi_1)$, $\eta_y = \sqrt{3/2}(\xi_3 - \xi_2)$ (ξ_i are just columns of P_3 and their linear combinations.) The orthogonal vectors are given by

$$\nu_1 = \sqrt{3/2}(\xi_1 - \xi_4) = \begin{pmatrix} 1 \\ 0 \\ -1/2 \\ -\sqrt{3}/2 \\ -1/2 \\ \sqrt{3}/2 \end{pmatrix}, \qquad \nu_2 = \sqrt{3/2}(\xi_2 + \xi_3) = \begin{pmatrix} 0 \\ 1 \\ \sqrt{3}/2 \\ -1/2 \\ -\sqrt{3}/2 \\ -1/2 \end{pmatrix}.$$

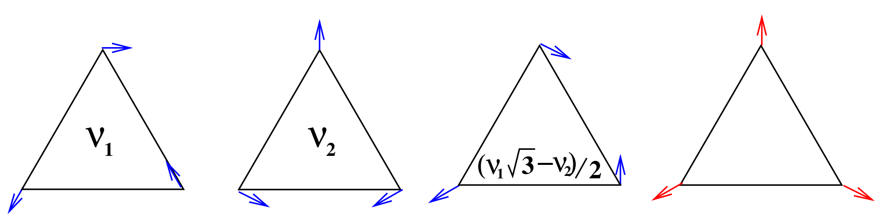


Figure 2.3: Modes of a molecule with D_3 symmetry. (B. Gutkin)

(B. Gutkin)

2.2 Discussion

2017-08-31 Michael Meehan <xmeehan@gatech.edu>, writes: When talking about the cosets of a subgroup we demonstrated multiplication between cosets with a specific example, but this wasn't leading to something along the lines of that the set of all left cosets of a subgroup (or the set of all the right cosets of a subgroup) form a group, correct? It didn't appear so in the example since the "unit" $\{E, A\}$ we looked appears to only have the properties of an identity with multiplication from one direction (the direction depending on if it is the set of left cosets or the set of right cosets). In the context of the lecture I think this point was related to Lagrange's theorem (although we didn't call it that) and I vaguely remember cosets being used in the proof of Lagrange's theorem but I wasn't connecting it today. Are we going to cover that in a future lecture?

2017-08-31 Predrag You are right - Lagrange's theorem (see the wiki) simply says the order of a subgroup has to be a divisor of the order of the group. We used cosets to partition elements of G to prove that. But what we really need cosets for is to define (see Dresselhaus *et al.* [1] Sect. 1.7) *Factor Groups* whose elements are cosets of a self-conjugate subgroup (click here). I will not cover that in a subsequent lecture, so please read up on it yourself.

2017-08-31 Michael Meehan You talked about the period of an element X, and said that that *period* is the *set*

$$\{E, X, \cdots, X^{n-1}\}, \tag{2.12}$$

where n is the *order* of the element X. I had thought that set was the subgroup generated by the element X and that the period of the element X was a synonym for the order of the element X? Is that incorrect?

2017-09-04 Predrag To keep things as simple as possible, in Thursday's lecture I followed Sect. 1.3 *Basic Definitions* of Dresselhaus *et al.* textbook [1], to the letter. In Def. 3 the *order* of an element X is the smallest n such that $X^n = E$, and they call the set (2.12) the *period* of X. I do not like that usage (and do not remember seeing it anywhere else). As you would do, in ChaosBook.org Chap. *Flips, slides and turns* I also define the smallest n to be the *period* of X and refer to the set (2.12) as the *orbit* generated by X. When we get to compact continuous groups, the orbit will be a (great) circle generated by a given Lie algebra element, and look more like what we usually think of as an orbit.

I am not using my own ChaosBook.org here, not to confuse things further by discussing both time evolution and its discrete symmetries. Here we focus on the discrete group only (typically spatial reflections and finite angle rotations)

References

[1] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[2] R. Mainieri and P. Cvitanović, "A brief history of chaos", in *Chaos: Classical and Quantum*, edited by P. Cvitanović, R. Artuso, R. Mainieri, G. Tanner, and G. Vattay (Niels Bohr Inst., Copenhagen, 2017).

[3] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

Exercises

2.1. **$G_x \subset G$.** The maximal set of group actions which maps a state space point x into itself,

$$G_x = \{g \in G : gx = x\}\,, \tag{2.13}$$

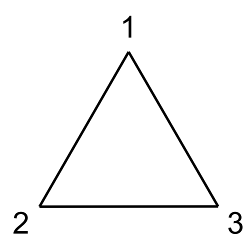
is called the *isotropy group* (or *stability subgroup* or *little group*) of x. Prove that the set G_x as defined in (2.13) is a subgroup of G.

2.2. **Transitivity of conjugation.** Assume that g_1, g_2, $g_3 \in G$ and both g_1 and g_2 are conjugate to g_3. Prove that g_1 is conjugate to g_2.

2.3. **Isotropy subgroup of gx.** Prove that for $g \in G$, x and gx have conjugate isotropy subgroups:

$$G_{gx} = g\, G_x\, g^{-1}$$

2.4. **D_3: symmetries of an equilateral triangle.** Consider group $D_3 \cong C_{3v}$, the symmetry group of an equilateral triangle:



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(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D_3.

(c) Find the classes of D_3 and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D_3. (continued as exercise 4.1)

2.5. **C_4-invariant potential.** Consider the Schrödinger equation for a particle moving in a two-dimensional bounding potential V, such that the spectrum is discrete. Assume that V is C_N-invariant, i.e., V remains invariant under the rotation R by the angle $2\pi/N$. It was explained in the lecture, that for $N = 3$ case, figure 2.4 (a), the spectrum of the system can be split into two sectors: $\{E_n^0\}$ non-degenerate levels corresponding to symmetric eigenfunctions $\phi_n(Rx) = \phi_n(x)$ and doubly degenerate levels $\{E_n^\pm\}$ corresponding to non-symmetric eigenfunctions $\phi_n(Rx) = e^{\pm 2\pi i/3}\phi_n(x)$.

Q 1 What is the spectral structure in the case of $N = 4$, figure 2.4 (b)? How many sectors appear and what are their degeneracies?

Q 2 What is the spectral structure for general N?

Q 3 A constant magnetic field normal to the $2D$ plane is added to V. How will it affect the spectral structure?

Q 4 (bonus question) Figure out the spectral structure if the symmetry group of potential is D_3 (also includes 3 reflections), figure 2.4 (c).

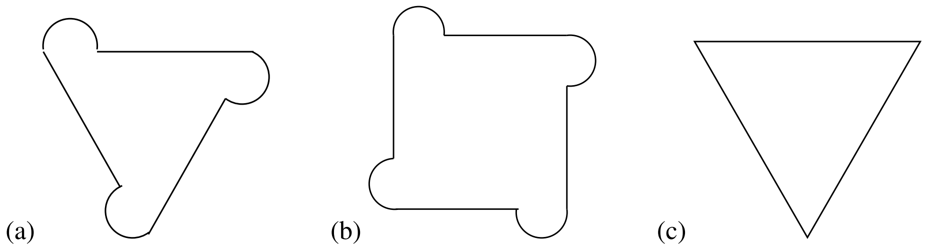

Figure 2.4: Hard wall potential with (a) symmetry C_3, (b) symmetry C_4, and (c) symmetry D_3.

(Boris Gutkin)

2.6. **Permutation of three objects.** Consider S_3, the group of permutations of 3 objects.

(a) Show that S_3 is a group.

(b) List the conjugacy classes of S_3?

(c) Give an interpretation of these classes if the group elements are substitution operations on a set of three objects.

(c) Give a geometrical interpretation in case of group elements being symmetry operations on equilateral triangle.

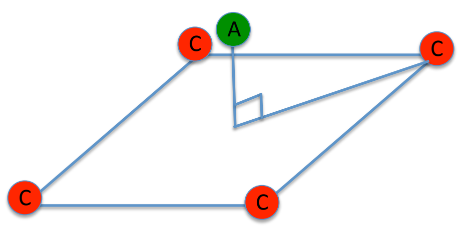

Figure 2.5: 4 identical particles of type C lie on the vertices of a square. In the center of the square, but out of the plane, is a particle of type A. (K. Y. Short)

2.7. **Arrangement of five particles.** Consider the arrangement of particles illustrated in figure 2.5: on each corner (vertex) of a rigid square lies a particle C; in the center of the square, but out of the plane on the z axis, is the particle A.

(a) What are the symmetries of this arrangement?

(b) Find its multiplication table.

(c) Find its subgroups.

(d) Determine the corresponding left and right cosets.

(e) Determine its conjugacy classes.

(f) Which subgroups are self-conjugate?

(g) Describe their factor groups.

(K. Y. Short)

2.8. **Three masses on a loop.** Three identical masses, connected by three identical springs, are constrained to move on a circle hoop as shown in figure 2.6. Find the normal modes. Hint: write down coupled harmonic oscillator equations, guess the form of oscillatory solutions. Then use basic matrix methods, i.e., find zeros of a characteristic determinant, find the eigenvectors, etc.. (K. Y. Short)

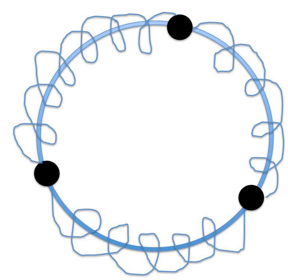

Figure 2.6: Three identical masses are constrained to move on a hoop, connected by three identical springs such that the system wraps completely around the hoop. Find the normal modes.