group theory - week 16

Wigner 3- and 6-j coefficients

Georgia Tech PHYS-7143

Homework HW16 due whenever - optional, not graded

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 16.1 *Gravity tensors*, part (a)	2 points
Exercise 16.1 *Gravity tensors*, part (b)	4 points
Exercise 16.1 *Gravity tensors*, part (c)	1 point
Exercise 16.1 *Gravity tensors*, part (d)	2 points
Exercise 16.1 *Gravity tensors*, part (e)	3 points
Exercise 16.1 *Gravity tensors*, part (f)	4 points
Exercise 16.1 *Gravity tensors*, part (g)	3 points
Exercise 16.1 *Gravity tensors*, part (h)	6 points

Bonus points

Exercise 16.1 *Gravity tensors*, part (i)	4 points
Exercise 16.1 *Gravity tensors*, part (j)	10 points

Total of 20 points = 100 % score.

2017-12-05 Predrag Lecture 30 Wigner 3- and 6-j coefficients

Excerpts from Predrag's monograph [2], fetch them here:

Background reading on groups, vector spaces, tensors, invariant tensors, invariance groups (my advice is to start with Sect. 5.1 *Couplings and recouplings*, then backtrack to these introductory sections as needed): Sect. 3.2 *Defining space, tensors, reps*, Sect. 3.3 *Invariants*, Sect. 4.1 *Birdtracks*, Sect. 4.2 *Clebsch-Gordan coefficients*, and Sect. 4.3 *Zero- and one-dimensional subspaces*.

The final result, discussed in the day's whiteboard-side chat, is invariant and highly elegant: any group-theoretical invariant quantity can be expressed in terms of Wigner 3- and 6-j coefficients: Sect. 5.1 *Couplings and recouplings*, Sect. 5.2 *Wigner* $3n$-j *coefficients*, and Sect. 5.3 *Wigner-Eckart theorem*.

The rest is just bedside reading, nothing technical: Sect. 4.8 *Irrelevancy of clebsches* and Sect. 4.9 *A brief history of birdtracks*.

Course finale: *Indiana Jones* video (click here).

16.1 Literature

The old fashioned atomic physics is fixated on $\mathrm{SO}(3)$ / $\mathrm{SU}(2)$, and it is too explicit, with too many bras and kets, too many square roots, too many deliriously complicated Clebsch-Gordan coefficients that you do not need, and has way too many labels, all of them eventually summed over in the final answer.

I wrote my book [2] *Group Theory - Birdtracks, Lie's, and Exceptional Groups* to teach you how to compute everything you need to compute, without ever writing down a single Clebsch-Gordan coefficient. There are two versions. There is a particle-physics / Feynman diagrams version that is index free, graphical and easy to use (at least for the low-dimensional irreps discussed in my book). The key insights are already in Wigner's book [5]: the content of symmetry is a set of invariant numbers that he calls $3n$-j's. Then there are various mathematical flavors (Weyl group on Cartan lattice, etc.), elegant, but perhaps to elegant to be computationally practical.

But it is nearly impossible to deprogram people from years of indoctrination in QM and EM classes. The professors have no time to learn new stuff, and students love manipulating their mu's and nu's.

References

[1] S. L. Adler, J. Lieberman, and Y. J. Ng, "Regularization of the stress-energy tensor for vector and scalar particles propagating in a general background metric", Ann. Phys. **106**, 279–321 (1977).

[2] P. Cvitanović, *Group Theory - Birdtracks, Lie's, and Exceptional Groups* (Princeton Univ. Press, Princeton, NJ, 2008).

[3] R. Penrose, "Applications of negative dimensional tensors", in *Combinatorial mathematics and its applications*, edited by D. J. J.A. Welsh (Academic, New York, 1971), pp. 221–244.

[4] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972).

[5] E. P. Wigner, *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra* (Academic, New York, 1931).

Exercises

16.1. **Gravity tensors.** In this problem we will apply diagrammatic methods ("birdtracks") to construct and count the numbers of independent components of the "irreducible rank-four gravity curvature tensors." However, any notation that works for you is OK, as long as you obtain the same irreps and their dimensions. The goal of this exercise (longish, as much of it is the recapitulation of the material covered in the book) is to give you basic understanding for how Young tableaux work for groups other than $U(n)$. We start with

Part 1 : **$U(n)$ Young tableaux decomposition.**

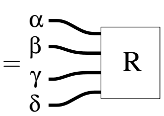
(a) The Riemann-Christoffel curvature tensor of general relativity has the following symmetries (see, for example, Weinberg [4] or the Riemann curvature tensor wiki):

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \qquad (16.1)$$

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \qquad (16.2)$$

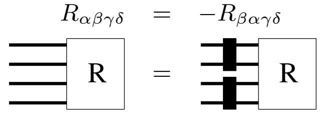
$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0\,. \qquad (16.3)$$

Introducing a birdtrack notation for the Riemann tensor

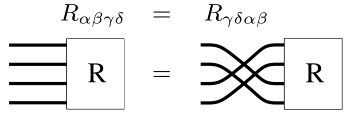
$R_{\alpha\beta\gamma\delta} =$  , (16.4)

check that we can state the above symmetries as

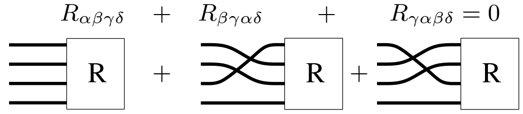
$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta}$$

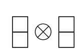
 , (16.5)

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$$

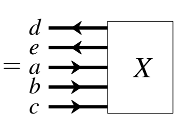
 , (16.6)

$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0$$

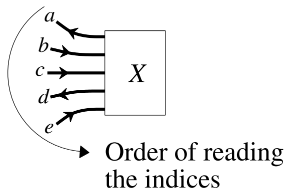
 $= 0\,.$ (16.7)

The first condition says that R lies in the  subspace.

Solution, in the index notation. First, a brief reminder about the notational conventions. Any invariant tensor can be drawn as a vector in the product space (a "generalized 1-vertex", a "tadpole"):

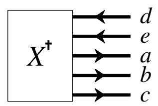
$X_\alpha = X^{abc}_{de} =$  . (16.8)

The indices are read in the *counterclockwise* order around the vertex:

$$X^{bce}_{ad} =$$ 

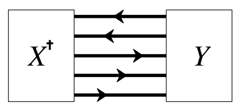
. (16.9)

For example, (16.7) is a diagrammatic representation of the cyclic permutations of the first 3 indexes of R_{abcd}.
Hermitian conjugation reverses the order of the indices, i.e., it transposes a diagram into its mirror image. For example, $X^\dagger$, the tensor conjugate to (16.9), is drawn as

$$X^\alpha = X^{ed}_{cba} =$$ 

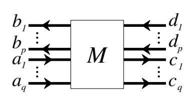
, (16.10)

and a contraction of tensors $X^\dagger$ and Y is drawn as

$$X^\alpha Y_\alpha = X^{b_p \ldots b_1}_{a_q \ldots a_2 a_1} Y^{a_1 a_2 \ldots a_q}_{b_1 \ldots b_p} =$$ 

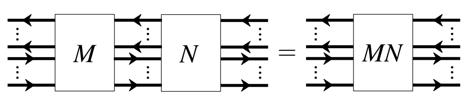
. (16.11)

We define the hermitian conjugation and matrices $\mathbf{M} : V^p \otimes \bar{V}^q \to V^p \otimes \bar{V}^q$ in the multi-index notation



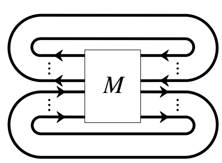
(16.12)

The matrix multiplication is drawn as



(16.13)

and the trace of a matrix as



. (16.14)

For orthogonal group SO(n) the indices can be raised and lowered at will (see (16.31)), so we can drop the arrows. However, as the hermitian conjugation now amounts to a transposition, we still need to keep track of the ordering of the indices (16.9) in order that the matrix multiplication (16.13) and the trace of a matrix (16.14) can be drawn in the plane.
The first two indices in (16.5) can be antisymmetrized by

$$A_{ab}{}^{b'a'} = \frac{1}{2}\left(\delta_a^{a'}\delta_b^{b'} - \delta_b^{a'}\delta_a^{b'}\right) \tag{16.15}$$

(or, equivalently in the case of SO(n), $A_{ab,b'a'}$). By the antisymmetry (16.1)

$$A_{ab}{}^{b'a'} R_{a'b'cd} = \frac{1}{2}\left(R_{abcd} - R_{bacd}\right) = R_{abcd} \tag{16.16}$$

By the index-pairs interchange (16.2),

$$
\begin{aligned}
E'{}_{abcd}{}^{d'c'b'a'} &= \delta_b^{d'}\delta_a^{c'}\delta_d^{b'}\delta_c^{a'} \\
(E'R)_{abcd} &= R_{cdab}
\end{aligned}
\qquad (16.17)
$$

the last two indices are also antisymmetric under

$$
A'{}_{cd}{}^{d'c'} = \frac{1}{2}\left(\delta_c^{c'}\delta_d^{d'} - \delta_c^{d'}\delta_d^{c'}\right), \qquad (16.18)
$$

and the whole Riemann tensor satisfies (16.5)

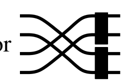
$$
\begin{aligned}
D_{abcd}{}^{d'c'b'a'}R_{a'b'c'd'} &= R_{abcd} \\
D_{abcd}{}^{d'c'b'a'} &= A_{cd}{}^{d'c'}A'{}_{ab}{}^{b'a'}.
\end{aligned}
\qquad (16.19)
$$

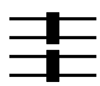
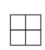
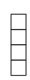
As A and A' act on distinct indices, they commute, $AA' = A'A$. Now combine the index-pairs interchange with the index-pair antisymmetrizations to construct a matrix that leaves tensors with Riemann tensor symmetries invariant:

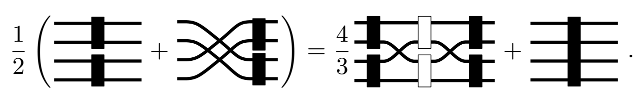
$$
\begin{aligned}
E_{abcd}{}^{d'c'b'a'} &= (E'D)_{abcd}{}^{d'c'b'a'} \\
E_{abcd}{}^{d'c'b'a'}R_{a'b'c'd'} &= R_{cdab}.
\end{aligned}
\qquad (16.20)
$$

The index-pairs interchange commutes with the index-pair antisymmetrizations $E'D = DE' = DE'D$.

(b) The second condition says that R lies in the  ↔  interchange-symmetric subspace.

Use the characteristic equation for 

to split  into the  and  irreps:


(16.21)

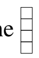
Solution The characteristic equation for E is

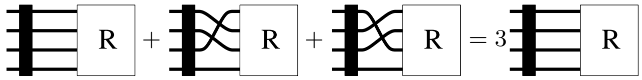
$$
E^2 = (DE'D)(DE'D) = D(E')^2D = D1D = D, \qquad (16.22)
$$

with the usual eigenvalues $\{\lambda_1, \lambda_2\} = \{1, -1\}$ and projection operators

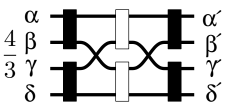
$$
P_\pm = \frac{1}{2}D(E' \pm 1)D. \qquad (16.23)
$$

$$
\begin{aligned}
(P_\pm)_{abcd}{}^{d'c'b'a'} = \; & \frac{1}{8}\left(\delta_a^{a'}\delta_b^{b'} - \delta_b^{a'}\delta_a^{b'}\right)\left(\delta_c^{c'}\delta_d^{d'} - \delta_c^{d'}\delta_d^{c'}\right) \\
& \left(\delta_b^{d'}\delta_a^{c'}\delta_d^{b'}\delta_c^{a'} \pm \delta_d^{d'}\delta_c^{c'}\delta_b^{b'}\delta_a^{a'}\right) \\
& \left(\delta_d^{d'}\delta_c^{c'}\delta_b^{b'}\delta_a^{a'} \pm \delta_d^{d'}\delta_c^{c'}\delta_b^{b'}\delta_a^{a'} \pm \delta_d^{d'}\delta_c^{c'}\delta_b^{b'}\delta_a^{a'} \pm \delta_d^{d'}\delta_c^{c'}\delta_b^{b'}\delta_a^{a'}\right).
\end{aligned}
\qquad (16.24)
$$

(c) Show that the third condition (16.7) says that R has no components in the  irrep:

 (16.25)

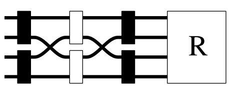
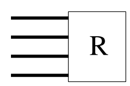
Hence, the symmetries of the Riemann tensor are summarized by the  irrep projection operator [3]:

$$(P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} = \frac{4}{3}$$  (16.26)

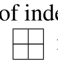
Solution

(d) Verify that the Riemann tensor is in the  subspace

$$(P_R R)_{\alpha\beta\gamma\delta} = (P_R)_{\alpha\beta\gamma\delta,}{}^{\delta'\gamma'\beta'\alpha'} R_{\alpha'\beta'\gamma'\delta'} = R_{\alpha\beta\gamma\delta}$$

$\frac{4}{3}$  =  . (16.27)

Solution

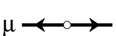
(e) Compute the number of independent components of the Riemann tensor $\mathbf{R}_{\alpha\beta\gamma\delta}$ by taking the trace of the  irrep projection operator:

$$d_R = \mathrm{tr}\,\mathbf{P}_R = \frac{n^2(n^2-1)}{12} \,. \tag{16.28}$$

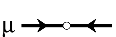
Solution

Part 2 : **SO**(n) **Young tableaux decomposition**

The Riemann tensor has the symmetries of the  irrep of U(n). However, gravity is also characterized by the symmetric tensor $g_{\alpha\beta}$, that reduces the symmetry to a local SO(n) invariance (more precisely SO$(1, n-1)$, but compactness is not important here). The extra invariants built from $g_{\alpha\beta}$'s decompose U(n) reps into sums of SO(n) reps. Orthogonal group SO(n) is the group of transformations that leaves invariant a symmetric quadratic form $(q, q) = g_{\mu\nu}q^\mu q^\nu$, with a primitive invariant rank-2 tensor:

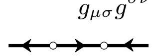
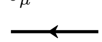
$g_{\mu\nu} = g_{\nu\mu} = \mu$  ν $\qquad \mu, \nu = 1, 2, \ldots, n\,.$ (16.29)

If (q, q) is an invariant, so is its complex conjugate $(q, q)^* = g^{\mu\nu}q_\mu q_\nu$, and

$g^{\mu\nu} = g^{\nu\mu} = \mu$  ν (16.30)

is also an invariant tensor. The matrix $A_\mu^\nu = g_{\mu\sigma}g^{\sigma\nu}$ must be proportional to unity, as otherwise its characteristic equation would decompose the defining n-dimensional rep. A convenient normalization is

$$g_{\mu\sigma}g^{\sigma\nu} = \delta_\mu^\nu$$

 =  . (16.31)

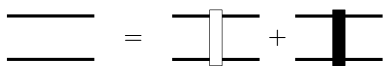
As the indices can be raised and lowered at will, nothing is gained by keeping the arrows. Our convention will be to perform all contractions with metric tensors with upper indices and omit the arrows and the open dots:

$$g^{\mu\nu} \equiv \mu \;\text{———}\; \nu\,. \tag{16.32}$$

The U(n) 2-index tensors can be decomposed into a sum of their symmetric and antisymmetric parts. Specializing to the subgroup SO(n), the rule is to lower all indices on all tensors, and the symmetrization projection operator is written as

$$\begin{aligned} S_{\mu\nu,\rho\sigma} &= g_{\rho\rho'}g_{\sigma\sigma'}S_{\mu\nu,}{}^{\rho'\sigma'} \\ &= \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right) \end{aligned}$$

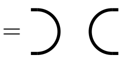
From now on, we drop all arrows and $g^{\mu\nu}$'s and write the decomposition into symmetric and antisymmetric parts as



$$g_{\mu\sigma}g_{\nu\rho} = \frac{1}{2}(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}) + \frac{1}{2}(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma})\,. \tag{16.33}$$

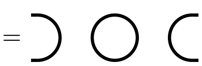
The new invariant tensor, specific to SO(n), is the index contraction:

$$\mathbf{T}_{\mu\nu,\rho\sigma} = g_{\mu\nu}g_{\rho\sigma}\,, \qquad \mathbf{T} = \tag{16.34}$$



Its characteristic equation

$$\mathbf{T}^2 = \quad = n\mathbf{T} \tag{16.35}$$

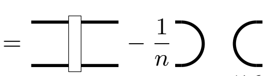


yields the trace and the traceless part projection operators. As $\mathbf{T}$ is symmetric, $S\mathbf{T} = \mathbf{T}$, only the symmetric subspace is reduced by this invariant.

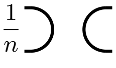
(f) Show that SO(n) 2-index tensors decompose into three irreps:

traceless symmetric:

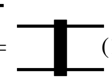
$$(P_2)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} + g_{\mu\rho}g_{\nu\sigma}\right) - \frac{1}{n}g_{\mu\nu}g_{\rho\sigma} = \tag{16.36}$$



singlet: $$(P_1)_{\mu\nu,\rho\sigma} = \frac{1}{n}g_{\mu\nu}g_{\rho\sigma} = \frac{1}{n} \tag{16.37}$$



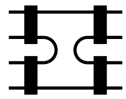
antisymmetric: $$(P_3)_{\mu\nu,\rho\sigma} = \frac{1}{2}\left(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}\right) = \tag{16.38}$$



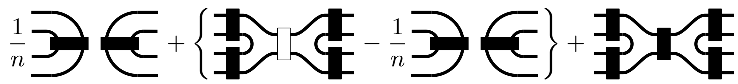
What are the dimensions of the three irreps?

Solution

(g) In the same spirit, the U(n) irrep  is decomposed by the SO(n) intermediate 2-index state invariant matrix

$\mathbf{Q} =$  . (16.39)

Show that the intermediate 2-index subspace splits into three irreducible reps by (16.36) – (16.38):

$\mathbf{Q} =$ 

$= \mathbf{Q}_0 + \mathbf{Q}_S + \mathbf{Q}_A \,.$ (16.40)

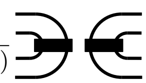
Show that the antisymmetric 2-index state does not contribute

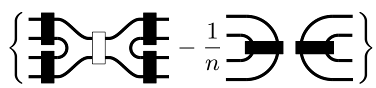
$$\mathbf{P_R Q}_A = 0\,. \qquad (16.41)$$

(Hint: The Riemann tensor is symmetric under the interchange of index pairs.)

Solution

(h) Fix the normalization of the remaining two projection operators by computing $\mathbf{Q}_S^2, \mathbf{Q}_0^2$:

$\mathbf{P}_0 = \frac{2}{n(n-1)}$  , (16.42)

$\mathbf{P}_S = \frac{4}{n-2}$  (16.43)

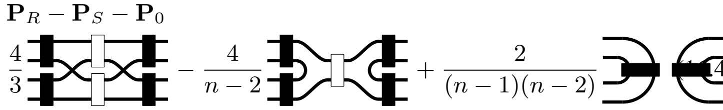
and compute their dimensions.
This completes the SO(n) reduction of the  U(n) irrep (16.27):

U(n)	$\to$	SO(n)				
	$\to$		+		+	$\circ$
$\mathbf{P}_R$	=	$\mathbf{P}_W$	+	$\mathbf{P}_S$	+	$\mathbf{P}_0$
$\frac{n^2(n^2-1)}{12}$	=	$\frac{(n+2)(n+1)n(n-3)}{12}$	+	$\frac{(n+2)(n-1)}{2}$	+	1

(16.44)

The projection operator for the SO(n) traceless  irrep is:

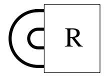
$\mathbf{P}_W = \mathbf{P}_R - \mathbf{P}_S - \mathbf{P}_0$

$\mathbf{P}_W =$  (16.45)

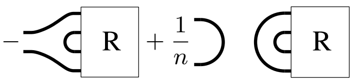
Solution

(i) The above three projection operators project out the standard, SO(n)-irreducible general relativity tensors:

Curvature scalar:

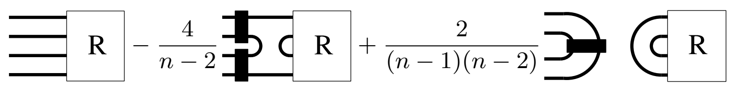
$R = -$  $= R^{\mu}{}_{\nu\mu}{}^{\nu}$ (16.46)

Traceless Ricci tensor:

$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = -$  (16.47)

Weyl tensor:

$$C_{\lambda\mu\nu\kappa} = (\mathbf{P}_W R)_{\lambda\mu\nu\kappa}$$

$=$ 

$$= R_{\lambda\mu\nu\kappa} + \frac{1}{n-2}(g_{\mu\nu}R_{\lambda\kappa} - g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\kappa}R_{\lambda\nu} + g_{\lambda\kappa}R_{\mu\nu}) - \frac{1}{(n-1)(n-2)}(g_{\lambda\kappa}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\kappa})R \,. \quad (16.48)$$

The numbers of independent components of these tensors are given by the dimensions of corresponding irreducible subspaces in (16.44).

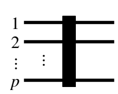
What is the lowest dimension in which the Ricci tensor contributes? the Weyl tensor contributes? Show that in 2, respectively 3 dimensions, we have

$$\begin{aligned} n=2: \quad R_{\lambda\mu\nu\kappa} &= (P_0 R)_{\lambda\mu\nu\kappa} = \tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R \\ n=3: \quad &= g_{\lambda\nu}R_{\mu\kappa} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu} - g_{\lambda\kappa}R_{\mu\nu} \\ &\quad -\tfrac{1}{2}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})R \quad . \end{aligned} \quad (16.49)$$

Solution

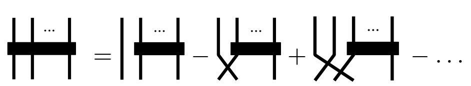
(j) The last example of this exercise is an application of birdtracks to general relativity index manipulations. The object is to find the characteristic equation for the Riemann tensor in *four dimensions*.

The antisymmetrization tensor $A_{a_1 a_2 \ldots,}{}^{b_p \ldots b_2 b_1}$ has nonvanishing components, only if all lower (or upper) indices differ from each other. If the defining dimension is smaller than the number of indices, the tensor A has no nonvanishing components:

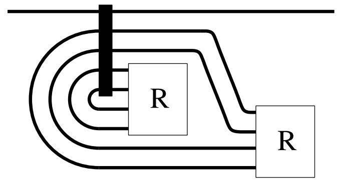


$= 0 \qquad \text{if } p > n\,.$ (16.50)

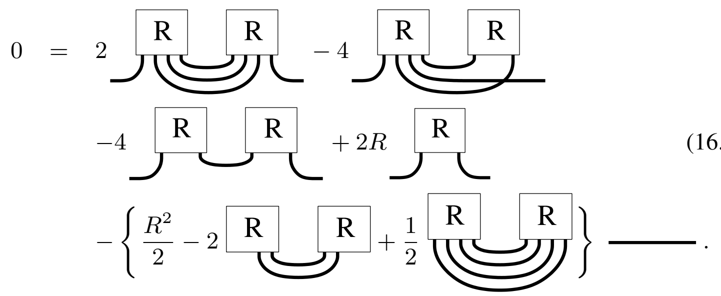
This identity implies that for $p > n$, not all combinations of p Kronecker deltas are linearly independent. A typical relation is the $p = n+1$ case

$0 =$  (16.51)

Contract (16.50) with two Riemann tensors:

$0 =$  , (16.52)

and obtain the characteristic equation by expanding with (16.51):



(16.53)

This identity has been used by Adler *et al.*, eq. (E2) in ref. [1].

Solution