group theory - week 15

Many particle systems. Young tableaux

Georgia Tech PHYS-7143

Homework HW15 due Tuesday, December 5

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 15.1 *Representations of* $SU(3)$	5 points
Exercise 15.2 *Young tableaux for* S_5	3 points

Bonus points

Exercise 15.3 *Young tableaux for* $SU(3)$	3 points
Exercise 15.4 *Irrep projection operators for unitary groups*	5 points

Total of 20 points = 100 % score.

2017-11-28 Predrag Lecture 28 Many particle systems. Young tableaux

Gutkin notes, Lect. 12 *Many particle systems*.

Excerpt from Predrag's monograph [4], fetch it here: Sect. 9.3 *Young tableaux*.

2017-11-30 Predrag Lecture 29 Young tableaux

Excerpts from Predrag's monograph [4], fetch them here: Sect. 2.2 *First example: SU*(n) (skim over casimirs and beyond: this example gives you a flavor of birdtracks computations, you do not need to work it out in detail), Sect. 6.1 *Symmetrization*, Sect. 6.2 *Antisymmetrization*, Sect. 9.1 *Two-index tensors*, Sect. 9.2 *Three-index tensors*, and Table 9.1.

Reading for this week: Sect. 9.3 *Young tableaux*.

15.1 Literature

The clearest current exposition and the most powerful irrep reduction of SU(n) is given in the triptych of papers by Judith Alcock-Zeilinger and her thesis adviser H. Weigart, University of Cape Town:

> *Simplification rules for birdtrack operators* [3],
> *Compact Hermitian Young projection operators* [2], and
> *Transition operators* [1].

References

[1] J. Alcock-Zeilinger and H. Weigert, "Transition operators", J. Math. Phys. **58**, 051702 (2016).

[2] J. Alcock-Zeilinger and H. Weigert, "Compact Hermitian Young projection operators", J. Math. Phys. **58**, 051702 (2017).

[3] J. Alcock-Zeilinger and H. Weigert, "Simplification rules for birdtrack operators", J. Math. Phys. **58**, 051701 (2017).

[4] P. Cvitanović, *Group Theory - Birdtracks, Lie's, and Exceptional Groups* (Princeton Univ. Press, Princeton, NJ, 2008).

Exercises

15.1. **Representations of SU(3).** Any irrep of SU(3) can be labeled $D(p,q)$ by its highest weight $\lambda = p\lambda_1 + q\lambda_2$, where $\lambda_{1,2}$ are the two fundamental weights.

(a) Find all irreps $D(p,q)$ of SU(3) with the dimensions less then 20 (see lecture notes for the dimensions of $D(p,q)$).

(b) Draw the lattice Λ generated by $\lambda_{1,2}$ and mark there all the weights v (i.e., lattice nodes) which belong to irrep. $D(3,0)$. Is $D(3,0)$ a real irrep?

(c) Consider product (reducible) representation $3 \otimes 3$, where $3 = D(1,0)$ is the fundamental irrep. Mark all the weights v on Λ which belong to $3 \otimes 3$. Using this find out decomposition of $3 \otimes 3$ into irreps:

$$3 \otimes 3 = \square \oplus \triangle, \qquad \square = ?, \qquad \triangle = ?$$

Hint: see lecture notes for similar exercise on $3 \otimes \bar{3}$.

(d) Using previous results find decomposition of $3 \otimes 3 \otimes 3$ into irreps.

(B. Gutkin)

15.2. **Young tableaux for S_5.**

(a) Draw all Young diagrams for the symmetric group S_5. How many irreducible representations has it? Which of the diagrams correspond to one-dimensional irreps?

(b) Find Young diagram corresponding to the irrep of S_5 with the largest dimension? Draw Young tableaux corresponding to this irrep/Young diagram. What is the dimension of this irrep?

(c) What are the dimensions of the remaining irreps?

(B. Gutkin)

15.3. **Young tableaux for SU(3).** Solve exercise 15.1 (c,d) by using Young tableaux.

Remark: If Young tableaux for SU(3) are not covered in the lectures, learn them yourself from, for example, *Group Theory Birdtracks, Lie's, and Exceptional Groups.* The resulting simple recipe with 0 explanation can be found, for example, here *C.G. Wohl*.

(B. Gutkin)

15.4. **Irrep projection operators for unitary groups.** Derive projection operators and dimensions for irreps of the Kronecker product of the defining and the adjoint reps of SU(n) listed in *Group Theory Birdtracks, Lie's, and Exceptional Groups*, Table 9.3. (Ignore "indices," we have not defined them.)