group theory - week 13

Simple Lie algebras; SU(3)

Georgia Tech PHYS-7143

Homework HW13 due Tuesday, November 21, 2017

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 13.1 *Root systems of simple Lie algebras*	5 points
Exercise 13.2 *Meson octet*	5 points

Bonus points

Exercise 13.3 *$SU(3)$ symmetry in 3D Harmonic Oscillator*	5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2017-11-14 Predrag Lecture 24 Representations of simple algebras

Gutkin notes, Lect. 10 *Representations of simple algebras, general construction. Application to SU(3)*, Sects. 1-4.

2017-11-16 Predrag Lecture 25 Cartan construction of $\mathrm{SU}(3)$ irreps

Gutkin notes, Lect. 10 *Representations of simple algebras, general construction. Application to SU(3)*, Sect. 5.

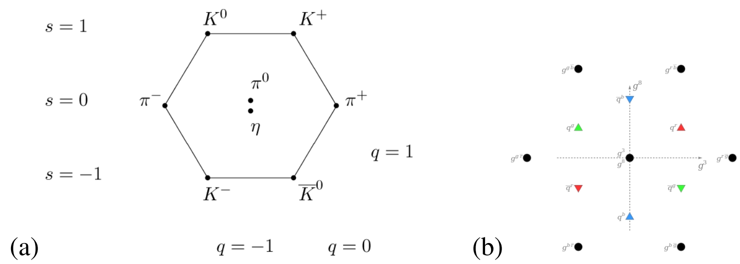

Figure 13.1: (a) The meson (pseudoscalars) octet. (b) The quark triplet, the anti-quark triplet and the gluon octet. (Wikipedia).

Exercises

13.1. **Root system of simple Lie algebras.**

a) Determine dimensions of Lie algebras $\mathfrak{so}(N), \mathfrak{su}(N)$ and dimensions of their Cartan subalgebras. What is the number of the positive roots for these Lie algebras?

b) Show that $N \times N$ diagonal matrices H_i with zero traces and uper/lower corner $N \times N$ matrices $E^{(a,b)}$ with the elements $E^{(a,b)}_{i,j} = \delta_{ia}\delta_{ib}$ provide Cartan-Weyl basis of $\mathfrak{su}(N)$. To put it differently, show that $E^{(a,b)}$ are eigenstates for adjoint representation of H_i's.

(B. Gutkin)

13.2. **Meson octet.** In Gutkin lecture notes, Lect. 11 *Strong interactions: flavor SU(3)*, the meson octet, figure 13.1 (a)

$$
\begin{aligned}
\Phi &= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}} & K^0 \\ K^- & \overline{K^0} & -\frac{2\eta}{\sqrt{6}} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} & \pi^+ & 0 \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & K^+ \\ 0 & 0 & K^0 \\ K^- & \overline{K^0} & 0 \end{pmatrix} + \frac{\eta}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix},
\end{aligned} \tag{13.1}
$$

is interpreted as arising from the adjoint representation of SU(3), i.e., the traceless part of the quark-antiquark $\mathbf{3} \otimes \overline{\mathbf{3}} = \mathbf{1} \oplus \mathbf{8}$ outer product (see figure 13.1 (b)),

$$
\begin{pmatrix} u\overline{u} & u\overline{d} & u\overline{s} \\ d\overline{u} & d\overline{d} & d\overline{s} \\ s\overline{u} & s\overline{d} & s\overline{s} \end{pmatrix} . \tag{13.2}
$$

where we have replaced in (13.1) the constituent $q \otimes \overline{q}$ combinations by the names of the elementary particles they build.

Given the quark quantum numbers

	Q	I	I_3	Y	B
u	2/3	1/2	1/2	1/3	1/3
d	-1/3	1/2	-1/2	1/3	1/3
s	-1/3	0	0	-2/3	1/3

verify the strangeness and charge assignments of figure 13.1 (a).

13.3. $SU(3)$ **symmetry in 3D Harmonic Oscillator.** The Hamiltonian of 3D isotropic harmonic oscillator is given by

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2m} + \frac{m\omega^2}{2} x_i^2 = \hbar\omega \sum_{i=1}^{3} (a_i^\dagger a_i + 1/2),$$

where $a_i = \sqrt{\frac{m\omega}{2\hbar}} x_i + i\sqrt{\frac{1}{2m\omega\hbar}} p_i$ is creation ($a_i^\dagger$ resp. annihilation) operator satisfying $[a_i, a_j^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$.

a) Show that $a_i \to U_{i,j} a_j$, with $U \in U(3)$ is a symmetry of the Hamiltonian. In other words isotropic 3D harmonic oscillator has $U(3)$ rather than $O(3)$ symmetry!

b) Calculate degeneracy of the n-th level $E_n = \omega\hbar(n + 3/2)$ of the oscillator.

c) By comparison of dimensions find out which representations of $SU(3)$ appear in the spectrum of harmonic oscillator.

(B. Gutkin)