group theory - week 12

Lorentz group; spin

Georgia Tech PHYS-7143

Homework HW12 due Tuesday, November 14, 2017

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 12.1 *Lorentz spinology*	5 points
Exercise 12.2 *Lorentz spin transformations*	5 points

Total of 10 points = 100 % score.

2017-11-07 Predrag Lecture 22
$\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$; Lorentz group

For $\mathrm{SO}(4) = \mathrm{SU}(2) \otimes \mathrm{SU}(2)$ see also birdtracks.eu chap. 10 *Orthogonal groups*, pp. 121-123; sect. 20.3.1 *$SO(4)$ or Cartan $A_1 + A_1$ algebra*

For Lorentz group, read Schwichtenberg [1] Sect. 3.7

2017-11-09 Predrag Lecture 23 $\mathrm{SO}(1,3)$; Spin

Schwichtenberg [1] Sect. 3.7

12.1 Spinors and the Lorentz group

A Lorentz transformation is any invertible real $[4 \times 4]$ matrix transformation Λ,

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{12.1}$$

which preserves the Lorentz-invariant Minkowski bilinear form $\Lambda^T \eta \Lambda = \eta$,

$$x^{\mu} y_{\mu} = x^{\mu} \eta_{\mu\nu} y^{\nu} = x^0 y^0 - x^1 y^1 - x^2 y^2 - x^3 y^3$$

with the metric tensor $\eta = diag(1, -1, -1, -1)$.

A contravariant four-vector $x^{\mu} = (x^0, x^1, x^2, x^3)$ can be arranged [2] into a Hermitian $[2 \times 2]$ matrix in $Herm(2, \mathbb{C})$ as

$$\underline{x} = \sigma_{\mu} x^{\mu} = \begin{pmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{pmatrix} \tag{12.2}$$

in the hermitian matrix basis

$$\sigma_{\mu} = \bar{\sigma}^{\mu} = (\mathbb{1}_2, \boldsymbol{\sigma}) = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)\,, \quad \bar{\sigma}_{\mu} = \sigma^{\mu} = (\mathbb{1}_2, -\boldsymbol{\sigma})\,, \tag{12.3}$$

with $\boldsymbol{\sigma}$ given by the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{12.4}$$

With the trace formula for the metric

$$\frac{1}{2} \mathrm{tr}\,(\sigma_{\mu} \bar{\sigma}_{\nu}) = \eta_{\mu\nu}\,, \tag{12.5}$$

the covariant vector x^{μ} can be recovered by

$$\frac{1}{2} \mathrm{tr}\,(\underline{x} \bar{\sigma}^{\mu}) = \frac{1}{2} \mathrm{tr}\,(x^{\nu} \sigma_{\nu} \bar{\sigma}^{\mu}) = x^{\nu} \eta_{\nu}{}^{\mu} = x^{\mu} \tag{12.6}$$

The Minkowski norm squared is given by

$$\det \underline{x} = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2 = x_{\mu} x^{\mu}\,, \tag{12.7}$$

and with (12.3)

$$\overline{x} = \begin{pmatrix} x^0 - x^3 & -x^1 + ix^2 \\ -x^1 - ix^2 & x^0 + x^3 \end{pmatrix} = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \tag{12.8}$$

the Minkowski scalar product is given by

$$x^\mu y_\mu = \frac{1}{2}\text{tr}(\underline{x}\,\overline{y}). \tag{12.9}$$

The *special linear group* $SL(2,\mathbb{C})$ in two complex dimensions is given by the set of all matrices Λ such that

$$SL(2,\mathbb{C}) = \{\Lambda \in GL(2,\mathbb{C}) \,|\, \det\Lambda = +1\}. \tag{12.10}$$

Let a matrix $\Lambda \in SL(2,\mathbb{C})$ act on $\underline{x} \in Herm(2,\mathbb{C})$ as

$$\underline{x} \mapsto \underline{x}' = \Lambda \underline{x} \Lambda^\dagger \tag{12.11}$$

where † denotes Hermitian conjugation. The Minkowski scalar product is preserved, $\det\underline{x}' = \det\underline{x}$. Thus $\underline{x}'$ can also be represented by a real linear combination of generalized Pauli matrices

$$\underline{x}' = \sigma_\mu x'^\mu \quad \text{with } x'_\mu x'^\mu = x_\mu x^\mu \tag{12.12}$$

and Λ explicitly acts as a Lorentz transformation (12.1), with $\Lambda^\mu_{\ \nu} = \frac{1}{2}\,\text{tr}\,(\bar{\sigma}^\mu \Lambda \sigma_\nu \Lambda^\dagger)$. The mapping is two-to-one, as two matrices $\pm\Lambda \in SL(2,\mathbb{C})$ generate the same Lorentz transformation $\Lambda\underline{x}\Lambda^\dagger = (-\Lambda)\underline{x}(-\Lambda)^\dagger$. This Λ belong to the proper orthochronous Lorentz group $SO^+(1,3)$, and it can be shown that $SL(2,\mathbb{C})$ is simply connected and is the double universal cover of the $SO^+(1,3)$.

Consider the fully antisymmetric Levi-Civita tensor $\varepsilon = -\varepsilon^{-1} = -\varepsilon^{\mathsf{T}}$ in two dimensions

$$\varepsilon = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{12.13}$$

This defines a *symplectic* (i.e., *skew-symmetric*) bilinear form $\langle u, v\rangle = -\langle v, u\rangle$ on two *spinors* u and v, elements of the two-dimensional complex vector (or spinor) space $\mathbb{C}^2$

$$u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix}, \quad v = \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \tag{12.14}$$

equipped with the symplectic form

$$\langle u, v\rangle = u^1 v^2 - u^2 v^1 = u^{\mathsf{T}} \varepsilon v. \tag{12.15}$$

This symplectic form is $SL(2,\mathbb{C})$-invariant

$$\langle u, v\rangle = u^{\mathsf{T}}\varepsilon v = \langle \Lambda u, \Lambda v\rangle = u^{\mathsf{T}}\Lambda^{\mathsf{T}}\varepsilon\Lambda v, \tag{12.16}$$

so one can interpret the group acting on spinors as $SL(2,\mathbb{C}) \cong Sp(2,\mathbb{C})$, the complex symplectic group in two dimensions

$$Sp(2,\mathbb{C}) = \{\Lambda \in GL(2,\mathbb{C}) \,|\, \Lambda^{\mathsf{T}}\varepsilon\Lambda = \varepsilon\}. \tag{12.17}$$

Summary. The group of Lorentz transformations of spinors is the group SL(2,C) of $[2\times2]$ complex matrices with determinant 1, i.e., the invariant tensor is the 2-index Levi-Civita ε_{AB}. The SL(2,C) matrices are parametrized by three complex dimensions and therefore six real ones (the matrices have four complex numbers and one complex constraint on the determinant). This matches the 6 dimensions of the group manifold associated with the Lorentz group $\mathrm{SO}(1,3)$.

Andrew M. Steane writes "A spinor is the most basic mathematical object that can be Lorentz-transformed." His *An introduction to spinors*, arXiv:1312.3824, might help you develop intuition about spinors.

12.2 Discussion

2017-10-17 Lin Xin Please explain the $M_{\mu\nu,\delta\rho}$ generators of $\mathrm{SO}(n)$.

2017-11-07 Predrag Let me know if you understand the derivation of Eqs. (4.51) and (4.52) in birdtracks.eu. Does that answer your question?

2017-11-07 Qimen Xu Please explain when one keeps track of the order of tensorial indices?

2017-11-07 Predrag In a tensor, upper, lower indices are separately ordered - and that order matters. The simplest example: if some indices form an antisymmetric pair, writing them in wrong order gives you a wrong sign. In a matrix representation of a group action, one has to distinguish between the "in" set of indices – the ones that get contracted with the initial tensor, and the "out" set of indices that label the tensor after the transformation. Only if the matrix is Hermitian the order does not matter. If you understand Eq. (3.22) in birdtracks.eu, you get it. Does that answer your question?

References

[1] J. Schwichtenberg, *Physics from Symmetry* (Springer, Berlin, 2015).

[2] E. Wigner, "On unitary representations of the inhomogeneous Lorentz group", Ann. Math. **40**, 149–204 (1939).

Exercises

12.1. **Lorentz spinology.**
Show that

(a)

$$x^2 = x_\mu x^\mu = \det \underline{x} \tag{12.18}$$

(b)

$$x_\mu y^\mu = \frac{1}{2}(\det(\underline{x}+\underline{y}) - \det(\underline{x}) - \det(\underline{y})) \tag{12.19}$$

(c)

$$x_\mu y^\mu = \frac{1}{2}\mathrm{tr}\;(\underline{x}\,\overline{y})\,, \tag{12.20}$$

where $\overline{y} = \bar{\sigma}_\mu y^\mu$

12.2. **Lorentz spin transformations.**
Let a matrix $\Lambda \in SL(2,\mathbb{C})$ act on hermitian matrix $\underline{x}$ as

$$\underline{x} \mapsto \underline{x}' = \Lambda \underline{x} \Lambda^\dagger\,. \tag{12.21}$$

(a) Check that $\underline{x}'$ is Hermitian, and the Minkowski scalar product (12.19) is preserved.

(b) Show that Λ explicitly acts as a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$.

(c) Show that the mapping from a $\Lambda \in SL(2,\mathbb{C})$ to the Lorentz transformation in $\mathrm{SO}(1,3)$ is two-to-one.

(d) Consider the Levi-Civita tensor $\epsilon = -\epsilon^{-1} = -\epsilon^{\mathrm{T}}$ in two dimensions,

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \tag{12.22}$$

and the associated symplectic form

$$\langle u, v\rangle = u^{\mathrm{T}}\varepsilon v = u^1 v^2 - u^2 v^1\,. \tag{12.23}$$

Show that this symplectic form is $SL(2,\mathbb{C})$-invariant

$$\langle u, v\rangle = u^{\mathrm{T}}\varepsilon v = \langle \Lambda u, \Lambda v\rangle = u^{\mathrm{T}}\Lambda^{\mathrm{T}}\varepsilon\Lambda v\,. \tag{12.24}$$