group theory - week 11

SU(2) and SO(3)

Georgia Tech PHYS-7143

Homework HW11 due Tuesday, November 7, 2017

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 11.1	*The characters of $SO(3)$ representations*	1 point
Exercise 11.2	*Lie algebra of $SO(4)$ and $SU(2) \otimes SU(2)$*	6 points
Exercise 11.5	SO(n) Clebsch-Gordan series for $V \otimes V$.	3 points

Bonus points

Exercise 11.3	*Real and pseudo-real representations of $SO(3)$*	4 points
Exercise 11.4	*Total spin of N particles*	5 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2017-10-31 Predrag Lecture 20 SU(2) and SO(3)

Gutkin notes, Lect. 9 *SU(2), SO(3) and their representations*, Sects. 1-3.2; sect. 11.1 *SU(2) – SO(3) correspondence* below.

2017-11-02 Predrag Lecture 21 SO(3) birdtracks

Birdtrack notation [1] is explained here.

You can fetch clippings on irreps of SU(n) and SO(n) from Predrag's monograph [1] here. Go through Sect. 2.2 *First example: SU(n)*, Sect. 6.1 *Symmetrization*, Sect. 6.2 *Antisymmetrization*, Sect. 9.1 *Two-index tensors*. Skim through Sect. 9.2 *Three-index tensors*, and Table 9.1. There is also a glimpse of a some birdtracking (still to be written up) in sect. 11.2 Irreps of SO(n).

Reading for the next week: Sect. 9.3 *Young tableaux*.

11.1 SU(2) – SO(3) correspondence

Notes by Kimberly Y. Short

Angular momentum $L = r \times p$ has three components, the operators that generate SU(2) and satisfy $[L_1, L_2] = iL_3$. If we define $e = L_1 + iL_2$, $f = L_1 - iL_2$, and $h = 2L_3$, then we have the following algebra:

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h \tag{11.1}$$

where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{11.2}$$

Matrices e and f act as the raising and lowering (also called 'ladder') operators $L_\pm$ in this representation. (The set $\{e, f, h\}$ forms an 'sl_2-triple'.)

We observe that there are $n^2 - 1 = 3$ such operators satisfying this algebra, which is the Lie algebra of SU(n), where $n = 2$. The eigenvalues of h are integers separated by 2, and the eigenvalues of L_3 must be half-integers separated by 1. Consequently, the representation with highest L_3 eigenvalue given by l must have dimension $2l + 1$ (note: $2l$ is λ_{max} for h).

Further, $L^2 = L \cdot L$ commutes with L_1, L_2, and L_3 and hence, by Schur's Lemma, $L^2 = \lambda \mathbb{I}$ in this representation, so every vector is an eigenvector of L^2. For example, we've seen in quantum mechanics,

$$L^2 \, Y_l^m = l(l+1)\hbar^2 \, Y_l^m \tag{11.3}$$

And since the spherical harmonics $Y_l^m(\theta, \phi)$ constitute an orthonormal basis of the Hilbert space of square-integrable functions, any vector can be expanded in a basis of $Y_l^m(\theta, \phi)$. $L_\pm$ act on Y_l^m in the following way:

$$L_\pm Y_l^m = \hbar\sqrt{l(l+1) - m(m \pm 1)} \, Y_l^{m\pm 1} \,. \tag{11.4}$$

An element of SU(2) can be written as

$$e^{i\sigma_j\alpha_j/2} \tag{11.5}$$

where σ_j is a Pauli matrix and α_j is a number. (The exponentiation of the Pauli matrices gives SU(2).) What is the importance of the $1/2$ factor in the argument of the exponential. First, consider a generic position vector $\boldsymbol{r} = x\hat{e}_i + y\hat{e}_j + z\hat{e}_k$. We may construct a matrix of the form

$$\begin{aligned}\sigma\cdot r &= \sigma_x x + \sigma_y y + \sigma_z z \\ &= \begin{pmatrix}0 & x\\ x & 0\end{pmatrix} + \begin{pmatrix}0 & -iy\\ iy & 0\end{pmatrix} + \begin{pmatrix}z & 0\\ 0 & -z\end{pmatrix} \\ &= \begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix}\end{aligned} \tag{11.6}$$

The determinant,

$$\det\begin{pmatrix}z & x-iy\\ x+iy & -z\end{pmatrix} = -(x^2+y^2+z^2) = -\boldsymbol{x}^2 \tag{11.7}$$

is an expression for the length of a vector.

Now consider a unitary transformation of this matrix. For example,

$$\begin{aligned}U(\sigma\cdot r)U^\dagger &= \sigma_x(\sigma\cdot r)\sigma_x \\ &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}z & x-iy\\ x-iy & z\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \\ &= \begin{pmatrix}-z & x+iy\\ x-iy & z\end{pmatrix}\end{aligned} \tag{11.8}$$

Taking this determinant, we find the same expression as before:

$$\det\begin{pmatrix}-z & x+iy\\ x-iy & z\end{pmatrix} = -(x^2+y^2+z^2) = -\boldsymbol{x}^2 \tag{11.9}$$

We observe that, like SO(3), SU(2) preserves the lengths of vectors.

The correspondence between SO(3) and SU(2) can be made more explicit. To see this, consider an SU(2) transformation on a two-component object called a *spinor* ψ where

$$\psi = \begin{pmatrix}\alpha\\ \beta\end{pmatrix}, \tag{11.10}$$

and

$$x = \frac{1}{2}(\beta^2-\alpha^2), \quad y = -\frac{i}{2}(\alpha^2+\beta^2), \quad z = \alpha\beta\,. \tag{11.11}$$

One may check that an SU(2) transformation on ψ is equivalent to an SO(3) transformation on $\boldsymbol{x}$. From this equivalence, one sees that an SU(2) transformation has three real parameters that correspond to the three rotation angles of SO(3). If we label the "angles" for the SU(2) transformation by α, β, and γ, we observe, for a "rotation" about $\hat{x}$

$$U_x(\alpha) = \begin{pmatrix} \cos\alpha/2 & i\sin\alpha/2 \\ i\sin\alpha/2 & \cos\alpha/2 \end{pmatrix} . \tag{11.12}$$

Likewise for an SU(2) transformation about $\hat{y}$:

$$U_y(\beta) = \begin{pmatrix} \cos\beta/2 & \sin\beta/2 \\ -\sin\beta/2 & \cos\beta/2 \end{pmatrix} \tag{11.13}$$

And for the final rotation, the SU(2) transformation about $\hat{z}$:

$$U_z(\gamma) = \begin{pmatrix} e^{i\gamma/2} & 0 \\ 0 & e^{-i\gamma/2} \end{pmatrix} \tag{11.14}$$

Compare these three matrices to the corresponding SO(3) rotation matrices:

$$R_x(\zeta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\zeta & \sin\zeta \\ 0 & -\sin\zeta & \cos\zeta \end{pmatrix}, \quad R_y(\phi) = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix}$$
$$R_z(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{11.15}$$

They're equivalent! Result: *Half the rotation angle generated by SU(2) corresponds to a rotation generated by SO(3).*

In this context, the eigenvalue equation for L_3 and for L^2 are differential equations whose solutions are the spherical harmonics Y_l^m which take the form

$$e^{im\phi}P_l^m(\cos\theta), \quad -l \le m \le l \tag{11.16}$$

in spherical coordinates and which determine the shape of electron orbitals and their probabilities to be found in a given region.

In quantum mechanics, the possible results of a measurement are determined by the possible eigenvalues of an operator. As such, the possible measurable values of the z-component of angular momentum correspond to the allowed values of L_3. The measurement outcomes are not arbitrary; the largest one, l, must be a half-integer, and there are $2l+1$ eigenvectors. Applying the lowering operator L_- one-by-one, we can find the possible outcomes to be $m \in \{l, l-1, ..., -l\}$. The angular dependence of the corresponding wave function goes as $\sim e^{im\phi}P_l^m(\cos\theta)$. In addition, higher values of l correspond to higher energy, so the different values of l correspond to different electron orbitals in order of increasing energy.

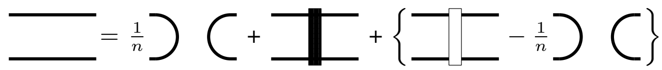
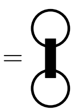
Young tableaux	$\square \times \square$	=	$\bullet$	+	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	+	$\square\square$
Dimensions	n^2	=	1	+	$\frac{n(n-1)}{2}$	+	$\frac{(n+2)(n-1)}{2}$
Projectors							

Table 11.1: SO(n) Clebsch-Gordan series for $V \otimes V$.

11.2 Irreps of SO(n)

The dimension of SO(n) is given by the trace of the adjoint projection operator:

$N = \mathrm{tr}\,\mathbf{P}_A =$  $= \frac{n(n-1)}{2} \,.$ (11.17)

Dimensions of the other reps are listed in table 11.1.

References

[1] P. Cvitanović, *Group Theory - Birdtracks, Lie's, and Exceptional Groups* (Princeton Univ. Press, Princeton, NJ, 2008).

Exercises

11.1. **The characters of SO(3) representations:** Show that for an irrep labeled by j, the character of a conjugacy class labeled by θ

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)} \tag{11.18}$$

can be obtained by taking the trace of $R_z^j(\theta)$. Verify that for $j = 1$ this character is the three dimensional special orthogonal representation character (10.6).

11.2. **Lie algebra of SO(4) and SU$(2) \otimes$ SU(2).** One particle Hamiltonian with a central potential has in general SO(3) symmetry group. It turns out, however, that for Coulomb potential the symmetry group is actually larger - SO(4), rather than SO(3). This explains why the energy level degeneracies in the hydrogen atom are anomalously large. So SO(4) and its representations are of a special importance in atomic physics.

(a) Show that the Lie algebra $\mathfrak{so}(4)$ of the group SO(4) is generated by real antisymmetric 4×4 matrices.

(b) What is the dimension of $\mathfrak{so}(4)$?

A natural basis in $\mathfrak{so}(4)$ is provided by antisymmetric matrices $M_{\mu\nu}$, $\mu, \nu \in 1, 2, 3, 4$, $\mu \neq \nu$, generators of SO(4) rotations which leave invariant the $\mu\nu$-plane. The elements of these matrices are given by

$$(M_{\mu\nu})_{ij} = \delta_{i\mu}\delta_{j\nu} - \delta_{j\mu}\delta_{i\nu} \tag{11.19}$$

(c) Check that these matrices satisfy the following commutation relationship:

$$[M_{ab}, M_{cd}] = M_{ad}\delta_{bc} + M_{bc}\delta_{ad} - M_{ac}\delta_{bd} - M_{bd}\delta_{ac}.$$

(d) Show that Lie algebras of the groups SO(4) and SU$(2) \times$ SU(2) are isomorphic. **Path:**

(d.i) Define matrices

$$J_k = \frac{1}{2}\varepsilon_{kij}M_{i,j}, \qquad K_k = M_{k4}, \quad k = 1, 2, 3$$

and

$$\mathcal{A}_k = \frac{1}{2}\left(J_k + K_k\right) \quad \text{and} \quad \mathcal{B}_k = \frac{1}{2}\left(J_k - K_k\right) .$$

(d.ii) Show that $\mathcal{A}$ and $\mathcal{B}$ satisfy the same commutation relations as two copies of $\mathfrak{su}(2)$.

(e) How does one construct irreps of $\mathfrak{so}(4)$ out of irreps of $\mathfrak{su}(2)$?

(f) Are groups SO(4) and SU$(2) \otimes$ SU(2) isomorphic to each other?

(B. Gutkin)

11.3. **Real and pseudo-real representations of $SO(3)$.** Recall (Gutkin notes, Lect. 4 *Representation Theory II*, Sect. 5 *5. Three types of representations*) that there are exist three types of representation which can be distinguished by the indicator:

$$\int_G d\mu(g)\chi_l(g^2) = \begin{cases} +1 & \text{real} \\ 0 & \text{complex} \\ -1 & \text{pseudo-real} \end{cases} . \tag{11.20}$$

Determine for which values of $l = 0, 1/2, 1, 3/2, 2\dots$ the representation D_l of $SO(3)$ is real or pseudo-real.

Hint: The characters and Haar measure (10.8) of $SO(3)$ are given by

$$\chi_l(g) = \frac{\sin\left(\left[l+\frac{1}{2}\right]\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}, \qquad d\mu(g) = \frac{d\theta}{\pi}\sin^2(\theta/2) \tag{11.21}$$

where θ is rotation angle for the group element g.

(B. Gutkin)

11.4. **Total spin of N particles.** Consider a system of four particles with spin $1/2$. Assuming that all (except spin) degrees of freedom are frozen the Hilbert space of the system is given by $V = V_{1/2} \otimes V_{1/2} \otimes V_{1/2} \otimes V_{1/2}$, with $V_{1/2}$ being two-dimensional space for each spin. $V = \oplus V_s$ can be decomposed then into different sectors V_s having the total spin s i.e., $\hat{S}^2 v = s(s+1)v$, for any $v \in V_s$. Here $\hat{S}^2 = (\sum_{i=1}^4 \hat{s}_i)^2$ and $\hat{s}_i = (\hat{s}_i^x, \hat{s}_i^y, \hat{s}_i^z)$ is spin operator for i-th particle.

(a) What are possible values s for the total spin of the system?

(b) Determine dimension of the subspace of V_0 with 0 total spin. In other words: how many times trivial representation enters into product:

$$D = D_{1/2} \otimes D_{1/2} \otimes D_{1/2} \otimes D_{1/2}\,? \tag{11.22}$$

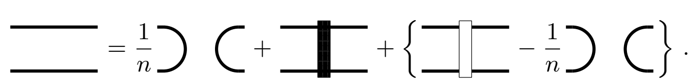
(c) What is the answer to the above questions for N spins?

Hint: it is convenient to use (11.21) to decompose D into irreps.

(B. Gutkin)

11.5. **SO(n) Clebsch-Gordan series for $V \otimes V$.**

(a) Show that the product of two n-dimensional reps of SO(n) decomposes into three irreps:


(11.23)

(b) Compute the dimensions of the three irreps.
(c) Which one is the adjoint one, and why? Hint: check the invariance condition.

11.6. **Splitting of degeneracies in a central potential.** Hamiltonian H_0 has rotational symmetry of $SO(3)$.

(a) What are the possible energy level degeneracies of H_0?

A weak perturbation V with a symmetry T_d of full tetrahedron group is added (e.g., V is a potential created by lattice of atoms with a symmetry of T_d).

(b) What will be the degeneracies of new Hamiltonian $H_0 + V$?

(c) Assuming that the total angular momentum of the system before the perturbation is $l = 2$. How the degeneracies of the corresponding energy level will be split after the perturbation is applied?

(B. Gutkin)

11.7. **Quadrupole transitions.**

a) Write $Q_1 = xy$, $Q_2 = zy$, $Q_3 = x^2 - y^2$ and $Q_4 = 2z^2 - x^2 - y^2$ as components of spherical tensor of rank 2. *Hint:* use spherical harmonics $Y_l^m(\theta, \varphi)$.

b) The last quantity Q_4 is known as quadrupole moment. What are the selection rules for transitions induced by Q_4 in a system with $SO(3)$ symmetry? In other words, for which m, l and k, j the transition rates:

$$P_{m,l \to k,j} \sim |\langle m\, l | Q_4 | j\, k \rangle|^2$$

are non-zero?

c) By using Wigner-Eckart theorem write down the relationship between $|\langle m\, l | Q_4 | j\, k \rangle|^2$ and $|\langle m\, l | Q_1 | j\, k \rangle|^2$ in terms of Clebsch-Gordan coefficients.

(B. Gutkin)