group theory - week 10

O(2) symmetry sliced

Georgia Tech PHYS-7143

Homework HW10 due Thursday, November 2, 2017

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 10.1 *Conjugacy classes of $SO(3)$* 2 points (+ 2 bonus points, if complete)
Exercise 10.2 *The character of $SO(3)$ 3-dimensional representation* 1 point
Exercise 10.4 *The orthonormality of $SO(3)$ characters* 2 point
Exercise 10.5 *$U(1)$ equivariance of two-modes system for finite angles* 3 points
Exercise 10.7 *$SO(2)$ or harmonic oscillator slice* 2 points

Bonus points
Exercise 10.6 *Integrate the two-modes system* 4 point
Exercise 10.8 *Invariant subspace of the two-modes system* 1 point
Exercise 10.9 *Slicing the two-modes system* 1 point
Exercise 10.10 *The symmetry reduced two-modes flow* (difficult) 6 points

Total of 10 points = 100 % score.

2017-10-24 Predrag Lecture 18 Lie groups, algebras Bridging the step from discrete to continuous compact groups: invariant integration measures, characters, character orthonormality and completeness relations.

Reading: ChaosBook.org Chap. *Continuous symmetry factorization* (last updated October 26, 2017), only Sect 26.1 *Compact groups*.

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Reading: sect. 10.3 *Two-modes* $SO(2)$*-equivariant flow*. For the long version, see ChaosBook.org Chap. *Relativity for cyclists* (last updated October 26, 2017), and ChaosBook.org Chap. *Slice & dice* (last updated October 26, 2017), Sect. 13.1 *Only dead fish go with the flow* to Sect. 13.5 *First Fourier mode slice*. This is difficult material, so it is OK if you do not get it this time around. None of this will be on the final - the main point is that once you face a nonlinear problem, nothing is easy - not even rotations on a circle.

10.1 Literature

C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, works out in full detail the representations and Haar measures for 1-dimensional Lie groups, and explains the difference between rotations and translations.

Chen, Ping and Wang [1] *Group Representation Theory for Physicists*, Sect 5.3 *Lie algebras* and Sect 5.4 *Finite transformations* work out several $\mathrm{SU}(2)$ and $\mathrm{O}(3)$ examples. Sects 5.5, 5.6 and 5.7 also merit a quick read.

In his group theory notes D. Vvedensky, chapter 8, sect. 8.3 *Axis–angle representation of proper rotations in three dimensions*, has a very nice discussion of the (10.2) parametrization of the $\mathrm{SO}(3)$ 3-dimensional group manifold: the parameter space corresponds to the interior of a sphere of radius π, and the over the classes of $\mathrm{SO}(3)$ is given by integral over spherical shells. In sect. 8.4 he derives the Haar measure (without calling it so).

In sect. 8.5 Vvedensky says: "For $\mathrm{SO}(2)$, we were able to determine the characters of the irreducible representations directly, i.e., without having to determine the basis functions of these representations. The structure of $\mathrm{SO}(3)$, however, does not allow for such a simple procedure, so we must determine the basis functions from the outset." That I disagree with; in birdtracks.eu sect. 15.1 *Reps of* $SU(2)$ I construct the irreps and label them by their Young tableaus with no recourse to spherical harmonics.

10.2 $\mathrm{SO}(3)$ character orthogonality

In 3 Euclidean dimensions, a rotation around z axis is given by the $\mathrm{SO}(2)$ matrix

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} = \exp\varphi \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \qquad (10.1)$$

An arbitrary rotation in $\mathbb{R}^3$ can be represented by

$$R_{\boldsymbol{n}}(\varphi) = e^{-i\varphi\,\boldsymbol{n}\cdot\boldsymbol{L}} \qquad \boldsymbol{L} = (L_1, L_2, L_3)\,, \tag{10.2}$$

where the unit vector $\boldsymbol{n}$ determines the plane and the direction of the rotation by angle φ. Here L_1, L_2, L_3 are the generators of rotations along x, y, z axes respectively,

$$L_1 = i\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{pmatrix}, \quad L_2 = i\begin{pmatrix} 0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix}, \quad L_3 = i\begin{pmatrix} 0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \tag{10.3}$$

with Lie algebra relations

$$[L_i, L_j] = i\varepsilon_{ijk}L_k\,. \tag{10.4}$$

All SO(3) rotations (10.2) by the same angle θ around different rotation axis $\boldsymbol{n}$ are conjugate to each other,

exercise 10.1

$$e^{i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}}e^{i\theta\boldsymbol{n}_1\cdot\boldsymbol{L}}e^{-i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}} = e^{i\theta\boldsymbol{n}_3\cdot\boldsymbol{L}}\,, \tag{10.5}$$

with $e^{i\phi\boldsymbol{n}_2\cdot\boldsymbol{L}}$ and $e^{-i\theta\boldsymbol{n}_2\cdot\boldsymbol{L}}$ mapping the vector $\boldsymbol{n}_1$ to $\boldsymbol{n}_3$ and back, so that the rotation around axis $\boldsymbol{n}_1$ by angle θ is mapped to a rotation around axis $\boldsymbol{n}_3$ by the same θ. The conjugacy classes of SO(3) thus consist of rotations by the same angle about all distinct rotation axes, and are thus labelled the angle θ. As the conjugacy class depends only on θ, the characters can only be a function of θ. For the 3-dimensional special orthogonal representation, the character is

exercise 10.2

$$\chi = 2\cos(\theta) + 1\,. \tag{10.6}$$

For an irrep labeled by j, the character of a conjugacy class labeled by θ is

exercise 10.3

$$\chi^{(j)}(\theta) = \frac{\sin(j + 1/2)\theta}{\sin(\theta/2)} \tag{10.7}$$

To check that these characters are orthogonal to each other, one needs to define the group integration over a parametrization of the SO(3) group manifold. A group element is parametrized by the rotation axis $\boldsymbol{n}$ and the rotation angle $\theta \in (-\pi, \pi]$, with $\boldsymbol{n}$ a unit vector which ranges over all points on the surface of a unit ball. Note however, that a π rotation is the same as a $-\pi$ rotation ($\boldsymbol{n}$ and $-\boldsymbol{n}$ point along the same direction), and the $\boldsymbol{n}$ parametrization of SO(3) is thus a 2-dimensional surface of a unit-radius ball with the opposite points identified.

The Haar measure for SO(3) requires a bit of work, here we just use note that after the integration over the solid angle (characters do not depend on it), the Haar measure is

$$dg = d\mu(\theta) = \frac{d\theta}{2\pi}(1 - \cos(\theta)) = \frac{d\theta}{\pi}\sin^2(\theta/2)\,. \tag{10.8}$$

With this measure the characters are orthogonal, and the character orthogonality theorems follow, of the same form as for the finite groups, but with the group averages replaced by the continuous, parameter dependant group integrals

exercise 10.4

$$\frac{1}{|G|}\sum_{g\in G} \to \int_G dg\,.$$

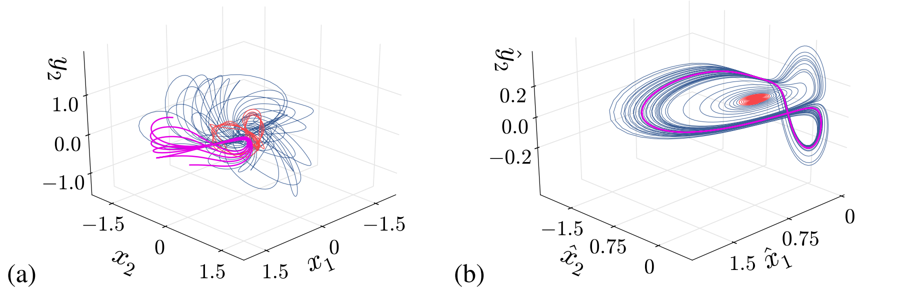

Figure 10.1: Two-modes flow before (a) and after (b) symmetry reduction by first Fourier mode slice. Here a long trajectory (red and blue) starting on the unstable manifold of the TW_1 (red), until it falls on to the strange attractor (blue) and the shortest relative periodic orbit $\overline{1}$ (magenta). Note that the relative equilibrium becomes an equilibrium, and the relative periodic orbit becomes a periodic orbit after the symmetry reduction.

The good news is that, as explained in ChaosBook.org Chap. *Relativity for cyclists* (and in *Group Theory - Birdtracks, Lie's, and Exceptional Groups* [2]), one never needs to actually explicitly construct a group manifold parametrizations and the corresponding Haar measure.

10.3 Two-modes SO(2)-equivariant flow

Consider the pair of U(1)-equivariant complex ODEs

$$\begin{aligned} \dot{z}_1 &= (\mu_1 - \mathrm{i}\, e_1)\, z_1 + a_1\, z_1 |z_1|^2 + b_1\, z_1 |z_2|^2 + c_1\, \overline{z}_1\, z_2 \\ \dot{z}_2 &= (\mu_2 - \mathrm{i}\, e_2)\, z_2 + a_2\, z_2 |z_1|^2 + b_2\, z_2 |z_2|^2 + c_2\, z_1^2\,, \end{aligned} \qquad (10.9)$$

with z_1, z_2 complex, and all parameters real valued.

This system is a generic example of a few-modes truncation of a Fourier representation of some physical flow, such as fluid dynamics convection flow, truncated in such a way that the model exhibits the same symmetries as the full original problem, while being drastically simpler to study. It is a merely a toy model with no physical interpretation, just like the iconic Lorenz flow. We use it to illustrate the effects of continuous symmetry on chaotic dynamics.

We refer to this toy model as the *two-modes* system. It belongs to the family of simplest ODE systems that we know that (a) have a continuous U(1) / SO(2), but no discrete symmetry (if at least one of $e_j \neq 0$). (b) models 'weather', in the same sense that Lorenz equation models 'weather', (c) exhibits chaotic dynamics, (d) can be easily visualized, in the dimensionally lowest possible setting required for chaotic dynamics, with the full state space of dimension $d = 4$, and the SO(2)-reduced dynamics taking place in 3 dimensions, and (e) for which the method of slices reduces the symmetry by a single global slice hyperplane.

The model has an unreasonably high number of parameters. After some experimentation we fix or set to zero various parameters, and in the numerical examples that follow, we settle for parameters set to

$$\begin{aligned} \mu_1 &= -2.8\,,\ \mu_2 = 1\,,\ e_1 = 0\,,\ e_2 = 1\,, \\ a_1 &= -1\,,\ a_2 = -2.66\,,\ b_1 = 0\,,\ b_2 = 0\,,\ c_1 = -7.75\,,\ c_2 = 1\,, \end{aligned} \tag{10.10}$$

unless explicitly stated otherwise. For these parameter values the system exhibits chaotic behavior. Experiment! If you find a more interesting behavior for some other parameter values, please let us know. The simplified system of equations can now be written as a 3-parameter $\{\mu_1, c_1, a_2\}$ two-modes system,

$$\begin{aligned} \dot{z}_1 &= \mu_1 z_1 - z_1|z_1|^2 + c_1 \overline{z}_1 z_2 \\ \dot{z}_2 &= (1-\mathrm{i})\, z_2 + a_2 z_2|z_1|^2 + z_1^2\,. \end{aligned} \tag{10.11}$$

In order to numerically integrate and visualize the flow, we recast the equations in real variables by substitution $z_1 = x_1 + i\,y_1$, $z_2 = x_2 + i\,y_2$. The two-modes system (10.9) is now a set of four coupled ODEs

exercise 10.6

$$\begin{aligned} \dot{x}_1 &= (\mu_1 - r^2)\, x_1 + c_1\,(x_1x_2 + y_1y_2)\,, \qquad r^2 = x_1^2 + y_1^2 \\ \dot{y}_1 &= (\mu_1 - r^2)\, y_1 + c_1\,(x_1y_2 - x_2y_1) \\ \dot{x}_2 &= x_2 + y_2 + x_1^2 - y_1^2 + a_2x_2r^2 \\ \dot{y}_2 &= -x_2 + y_2 + 2\,x_1y_1 + a_2y_2r^2\,. \end{aligned} \tag{10.12}$$

Try integrating (10.12) with random initial conditions, for long times, times much beyond which the initial transients have died out. What is wrong with this picture? Figure 10.3 (a) is a mess. As we show here, the attractor is built up by a nice 'stretch & fold' action, hidden from the view by the continuous symmetry induced drifts. That is fixed by 'quotienting' model's SO(2) symmetry, and reducing the dynamics to a 3-dimensional symmetry-reduced state space, figure 10.3 (b).

exercise 10.7
exercise 10.8
exercise 10.9

References

[1] J.-Q. Chen, J. Ping, and F. Wang, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).

[2] P. Cvitanović, *Group Theory - Birdtracks, Lie's, and Exceptional Groups* (Princeton Univ. Press, Princeton, NJ, 2008).

Exercises

10.1. **Conjugacy classes of SO(3):** Show that all SO(3) rotations (10.2) by the same angle θ around any rotation axis $\boldsymbol{n}$ are conjugate to each other:

$$e^{i\phi \boldsymbol{n}_2 \cdot \boldsymbol{L}} e^{i\theta \boldsymbol{n}_1 \cdot \boldsymbol{L}} e^{-i\phi \boldsymbol{n}_2 \cdot \boldsymbol{L}} = e^{i\theta \boldsymbol{n}_3 \cdot \boldsymbol{L}} \tag{10.13}$$

Check this for infinitesimal ϕ, and argue that from that it follows that it is also true for finite ϕ. Hint: use the Lie algebra commutators (10.4).

10.2. **The character of SO(3) 3-dimensional representation:** Show that for the 3-dimensional special orthogonal representation (10.2), the character is

$$\chi = 2\cos(\theta) + 1\,. \tag{10.14}$$

Hint: evaluating the character explicitly for $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$.

10.3. **The characters of SO(3) representations:** Show that for an irrep labeled by j, the character of a conjugacy class labeled by θ can be obtained by taking the trace of $R_z^j(\theta)$, and that the character is

$$\chi^{(j)}(\theta) = \frac{\sin(j+1/2)\theta}{\sin(\theta/2)}\,. \tag{10.15}$$

Verify that for $j = 1$ this character is the three dimensional special orthogonal representation character (10.14).

10.4. **The orthonormality of SO(3) characters:** Verify that given the Haar measure (10.8), the characters (10.15) are orthogonal:

$$\langle \chi(j) | \chi(j') \rangle = \int_G dg\, \chi^{(j)}(g^{-1})\, \chi^{(j')}(g) = \delta_{jj'}\,. \tag{10.16}$$

10.5. **U(1) equivariance of two-modes system for finite angles:** Show that the vector field in two-modes system (10.9) is equivariant under (9.1), the unitary group U(1) acting on $\mathbb{R}^4 \cong \mathbb{C}^2$ as the $k = 1$ and 2 modes:

$$g(\theta)(z_1, z_2) = (e^{i\theta} z_1, e^{i2\theta} z_2)\,, \quad \theta \in [0, 2\pi)\,. \tag{10.17}$$

10.6. **Integrate the two-modes system:** Integrate (10.12) and plot a long trajectory of two-modes in the 4d state space, (x_1, y_1, y_2) projection, as in figure 10.3 (a). To save you time (typing in (10.12) is tedious), we have prepared for you python code, and online graded problem set here. If you do this exercise, please get started early, in order to make sure that the autograder is working, and forward to us the grades that you receive from the autograder.

10.7. **SO(2) or harmonic oscillator slice:** Construct a moving frame slice for action of SO(2) on $\mathbb{R}^2$

$$(x, y) \mapsto (x\cos\theta - y\sin\theta,\; x\sin\theta + y\cos\theta)$$

by, for instance, the positive y axis: $x = 0$, $y > 0$. Write out explicitly the group transformation that brings any point back to the slice. What invariant is preserved by this construction?

10.8. **Invariant subspace of the two-modes system:** Show that $(0, 0, x_2, y_2)$ is a flow invariant subspace of the two-modes system (10.12), i.e., show that a trajectory with the initial point within this subspace remains within it forever.

10.9. **Slicing the two-modes system:** Choose the simplest slice template point that fixes the 1. Fourier mode,

$$\hat{x}' = (1, 0, 0, 0)\,. \tag{10.18}$$

(a) Show for the two-modes system (10.12), that the velocity within the slice, and the phase velocity along the group orbit are

$$\hat{v}(\hat{x}) = v(\hat{x}) - \dot{\phi}(\hat{x})t(\hat{x}) \tag{10.19}$$

$$\dot{\phi}(\hat{x}) = -v_2(\hat{x})/\hat{x}_1 \tag{10.20}$$

(b) Determine the chart border (the locus of point where the group tangent is either not transverse to the slice or vanishes).

(c) What is its dimension?

(d) What is its relation to the invariant subspace of exercise 10.8?

(e) Can a symmetry-reduced trajectory cross the chart border?

10.10. **The symmetry reduced two-modes flow:** Pick an initial point $\hat{x}(0)$ that satisfies the slice condition for the template choice (10.18) and integrate (10.19) & (10.20). Plot the three dimensional slice hyperplane spanned by (x_1, x_2, y_2) to visualize the symmetry reduced dynamics. Does it look like figure 10.3 (b)?