group theory - week 9

Continuous groups

Georgia Tech PHYS-7143

Homework HW9 due Thursday, March 17, 2016

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 9.1 *Irreps of* $SO(2)$	2 points
Exercise 9.2 *Reduction of product of two* $SO(2)$ *irreps*	1 point
Exercise 9.3 *Irreps of* $O(2)$	2 points
Exercise 9.4 *Reduction of product of two* $O(2)$ *irreps*	1 point
Exercise 9.5 *A fluttering flame front*	4 points

Bonus points
Exercise 9.6 $O(2)$ *fundamental domain* (difficult) 10 points

Total of 10 points = 100 % score.

2016-03-08 Predrag Lecture 17 Continuous groups

This lecture is not taken from any particular book, it's about basic ideas of how one goes from finite groups to the continuous ones that any physicist should know. The main idea comes from discrete groups. We have worked one example out in week 2, the discrete Fourier transform of example 2.4 *Projection operators for cyclic group C_N*. The cyclic group C_N is generated by the powers of the rotation by $2\pi/N$, and in general, in the $N \to \infty$ limit one only needs to understand the algebra of T_ℓ, generators of infinitesimal transformations, $D(\theta) = 1 + i \sum_\ell \theta_\ell T_\ell$. They turn out to be derivatives.

2016-03-10 Predrag Lecture 18 Lie groups. Matrix representations

The $N \to \infty$ limit of C_N gets you to the continuous Fourier transform as a representation of $U(1) \simeq SO(2)$, but from then on this way of thinking about continuous symmetries gets to be increasingly awkward. So we need a fresh restart; that is afforded by matrix groups, and in particular the unitary group $U(n) = U(1) \otimes SU(n)$, which contains all other compact groups, finite or continuous, as subgroups.

Reading: Chen, Ping and Wang [1] *Group Representation Theory for Physicists*, Sect 5.2 *Definition of a Lie group, with examples*.

Reading: C. K. Wong *Group Theory* notes, Chap 6 *1D continuous groups*, Sects. 6.1-6.3 Irreps of $SO(2)$. In particular, note that while geometrically intuitive representation is the set of rotation $[2\times 2]$ matrices, they split into pairs of 1-dimensional irreps. Also, not covered in the lectures, but worth a read: Sect. 6.6 completes discussion of Fourier analysis as continuum limit of cyclic groups C_n, compares $SO(2)$, discrete translations group, and continuous translations group.

References

[1] J.-Q. Chen, J. Ping, and F. Wang, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).

Exercises

9.1. **Irreps of SO(2).** Matrix

$$T = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \tag{9.1}$$

is the generator of rotations in a plane.

(a) Use the method of projection operators to show that for rotations in the kth Fourier mode plane, the irreducible $1D$ subspaces orthonormal basis vectors are

$$\mathbf{e}^{(\pm k)} = \frac{1}{\sqrt{2}}\left(\pm\mathbf{e}_1^{(k)} - i\,\mathbf{e}_2^{(k)}\right) .$$

How does T act on $\mathbf{e}^{(\pm k)}$?

(b) What is the action of the $[2\times2]$ rotation matrix

$$D^{(k)}(\theta) = \begin{pmatrix}\cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta\end{pmatrix} , \qquad k = 1, 2, \cdots$$

on the $(\pm k)$th subspace $\mathbf{e}^{(\pm k)}$?

(c) What are the irreducible representations characters of SO(2)?

9.2. **Reduction of a product of two SO(2) irreps.** Determine the Clebsch-Gordan series for SO(2). Hint: Abelian group has 1-dimensional characters. Or, you are just multiplying terms in Fourier series.

9.3. **Irreps of O(2).** O(2) is a group, but not a Lie group, as in addition to continuous transformations generated by (9.1) it has, as a group element, a parity operation

$$\sigma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

which cannot be reached by continuous transformations.

(a) Is this group Abelian, i.e., does T commute with $R(k\theta)$? Hint: evaluate first the $[T, \sigma]$ commutator and/or show that $\sigma D^{(k)}(\theta)\sigma^{-1} = D^{(k)}(-\theta)$.

(b) What are the equivalence classes of this group?

(c) What are irreps of O(2)? What are their dimensions?

Hint: O(2) is the $n \to \infty$ limit of D_n, worked out in exercise 4.4 *Irreducible representations of dihedral group D_n*. Parity σ maps an SO(2) eigenvector into another eigenvector, rendering eigenvalues of any O(2) commuting operator degenerate. Or, if you really want to do it right, apply Schur's first lemma to improper rotations

$$R'(\theta) = \begin{pmatrix}\cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta\end{pmatrix}\sigma = \begin{pmatrix}\cos k\theta & \sin k\theta \\ \sin k\theta & -\cos k\theta\end{pmatrix}$$

to prove irreducibility for $k \neq 0$.

(d) What are irreducible characters of O(2)?

(e) Sketch a fundamental domain for O(2).

9.4. **Reduction of a product of two O(2) irreps.** Determine the Clebsch-Gordan series for O(2), i.e., reduce the Kronecker product $D^{(k)} \bigotimes D^{(\ell)}$.

9.5. **A fluttering flame front.**

(a) Consider a linear partial differential equation for a real-valued field $u = u(x,t)$ defined on a periodic domain $u(x,t) = u(x+L,t)$:

$$u_t + u_{xx} + \nu u_{xxxx} = 0 , \qquad x \in [0, L] . \tag{9.2}$$

In this equation $t \geq 0$ is the time and x is the spatial coordinate. The subscripts x and t denote partial derivatives with respect to x and t: $u_t = \partial u/d\partial$, u_{xxxx} stands for the 4th spatial derivative of $u = u(x,t)$ at position x and time t. Consider the form of equations under coordinate shifts $x \to x + \ell$ and reflection $x \to -x$. What is the symmetry group of (9.2)?

(b) Expand $u(x,t)$ in terms of its SO(2) irreducible components (hint: Fourier expansion) and rewrite (9.2) as a set of linear ODEs for the expansion coefficients. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(c) Expand $u(x,t)$ in terms of its O(2) irreducible components (hint: Fourier expansion) and rewrite (9.2) as a set of linear ODEs. What are the eigenvalues of the time evolution operator? What is their degeneracy?

(d) Interpret $u = u(x,t)$ as a 'flame front velocity' and add a quadratic nonlinearity to (9.2),

$$u_t + \frac{1}{2}(u^2)_x + u_{xx} + \nu u_{xxxx} = 0\,, \qquad x \in [0, L]\,. \tag{9.3}$$

This nonlinear equation is known as the Kuramoto-Sivashinsky equation, a baby cousin of Navier-Stokes. What is the symmetry group of (9.3)?

(e) Expand $u(x,t)$ in terms of its O(2) irreducible components (see exercise 9.3) and rewrite (9.3) as an infinite tower of coupled nonlinear ODEs.

(f) What are the degeneracies of the spectrum of the eigenvalues of the time evolution operator?

9.6. **O(2) fundamental domain.** You have C_2 discrete symmetry generated by flip σ, which tiles the space by two tiles.

- Is there a subspace invariant under this C_2? What form does the tower of ODEs take in this subspace?
- How would you restrict the flow (the integration of the tower of coupled ODEs) to a fundamental domain?