group theory - week 6

For fundamentalists

Georgia Tech PHYS-7143

Homework HW6 due Tuesday, February 23, 2016

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort
== if you are LaTeXing, here is the source code

Exercise 6.1 *3-disk symbolic dynamics*	2 points
Exercise 6.2 *Reduction of 3-disk symbolic dynamics to binary*	3 points
Exercise 6.3 *3-disk fundamental domain cycles*	2 points
Exercise 6.4 C_2*-equivariance of Lorenz system*	3 points

Bonus points
Exercise 6.5 *Proto-Lorenz system* 10 points

Total of 10 points = 100 % score. Extra points accumulate, can help you later if you miss a few problems.

2016-02-16 Predrag Lecture 11 Fundamental domain

The three disk example. Read ChaosBook.org Chapter 10 *Flips, slides and turns*. You already know much of the material covered in the text, so best to go straight to Example 10.2 *3-disk game of pinball - symmetry-related orbits*, Example 10.7 *Subgroups, cosets of D_3*, Example 11.4 *3-disk game of pinball - cycle symmetries*, Example 11.8 *3-disk game of pinball in the fundamental domain*, and then work your way backward, if there is something you do not understand off the bat..

2016-02-18 Boris Lecture 12 Problem solving session

Lorenz flow example 6.2. Read ChaosBook.org Chapter 11 *World in a mirror*. Maybe start with Example 10.6 *Equivariance of the Lorenz flow*, Example 11.5 *Desymmetrization of Lorenz flow*, and then work your way back if needed.

As we have used this lecture to work out exercise 5.1 *Vibration modes of CH_4*, coming Tuesday we will have the famed and much disliked brick & mortar "flipped classroom:" we expect you to complete the reading and the homework for this week, augmented by - if you find that helpful - by 'live' online blackboard lectures: click here.

6.1 Literature

2016-02-18 Predrag Heilman and Strichartz [1] *Homotopies of Eigenfunctions and the Spectrum of the Laplacian on the Sierpinski Carpet* is not an obvious read for us, but they compute a spectrum on a square domain, and we might have to be mindful of it: " Since all of our domains are invariant under the D_3 symmetry group, we can simplify the eigenfunction computations by reducing to a fundamental domain. On this domain we impose appropriate boundary conditions according to the rep-resentation type. For the 1-dimensional representation, we restrict to the sector $0 \leq \theta \leq \pi/4$. Recall that the functions will extend evenly when reflected about $\theta = 0$ in the 1++ and 1– cases, and oddly in the 1-+ and 1+- cases. Note that performing an even extension across a ray is equivalent to imposing Neumann boundary conditions on that ray. Similarly, the odd extension is equivalent to Dirichlet conditions. For the 2-dimensional representation our fundamental domain is the sector $0 \leq \theta \leq \pi/2$, and we impose Neumann boundary conditions on the ray $\theta = 0$ and Dirichlet conditions on the ray $\theta = \pi/2$. Notice that our fundamental domains are simply connected. Therefore, as in Theorem 6.4 of ref. [3], we should expect no two-dimensional eigenspaces along this homotopy (for each individual symmetry family). "

This seems to be saying that one gets the 2-dimensional representation by doubling the fundamental domain and mixing boundary conditions. Do you understand that? I have looked at Theorem 6.4 of ref. [3] and am clueless.

2016-02-21 Boris Here is my present understanding of the fundamental domains issue: If you want simple boundary conditions like Dirichlet or Neumann you stick to 1d representations only. They connect eigenfunction to itself at the fundamental domain boundaries – otherwise you would need to connect pair of functions (would be something like boundary conditions for spinor in case of 2d representations.) So what you do is the following: take the largest abelian subgroup $Z_2 \times Z_2$ (for D_4) and split its spectrum with respect to its fundamental domain defined as 1/4 of the square (twice the fundamental domain of the full group). Then your see that Dirichlet-Dirichlet and Neumann-Neumann Hamiltonians still have Z_2 symmetry so your split them further into the Hamiltonians of the 1/8 fundamental domain. But Dirichlet-Neumann remains 1/4th of the square.

I did not get what is the connection with the ref. [3]. This Theorem just says that you can find a path such that no degeneracies occur in between. Because of avoided crossings one can actually expect that a typical path (it should avoid symmetric configurations) has no degeneracies for non-integrable systems.

2016-02-22 Predrag Your argument is in the spirit of Harter's class operators construction of higher-dimensional representations by using particular chains of subgroups, but I am not able to visualize how that larger fundamental domain (of the lower-order subgroup) folds back into the small fundamental domain of the whole group. How I think of the fundamental domain is explained in my online lectures, Week 14, in particular the snippet Regular representation of permuting tiles. Unfortunately - if I had more time, that would have been shorter, this goes on and on, Week 15, lecture *29. Discrete symmetry factorization*, and by the time the dust settles, I have the symmetry factorization of the determinants that we need, but I do not have a gut feeling for the boundary conditions that you do, when it comes to higher-dimensional irreps. So you'll have to explain that to us...

Copied here are a few snippets from this week's lecture notes, needed here just because exercises refer to them - read the full lecture notes instead.

Definition: Flow invariant subspace. A typical point in fixed-point subspace $\mathcal{M}_H$ moves with time, but, due to equivariance

$$f(gx) = gf(x)\,, \tag{6.1}$$

its trajectory $x(t) = f^t(x)$ remains within $f(\mathcal{M}_H) \subseteq \mathcal{M}_H$ for all times,

$$hf^t(x) = f^t(hx) = f^t(x)\,, \quad h \in H\,, \tag{6.2}$$

i.e., it belongs to a *flow invariant subspace*. This suggests a systematic approach to seeking compact invariant solutions. The larger the symmetry subgroup, the smaller $\mathcal{M}_H$, easing the numerical searches, so start with the largest subgroups H first.

We can often decompose the state space into smaller subspaces, with group acting within each 'chunk' separately:

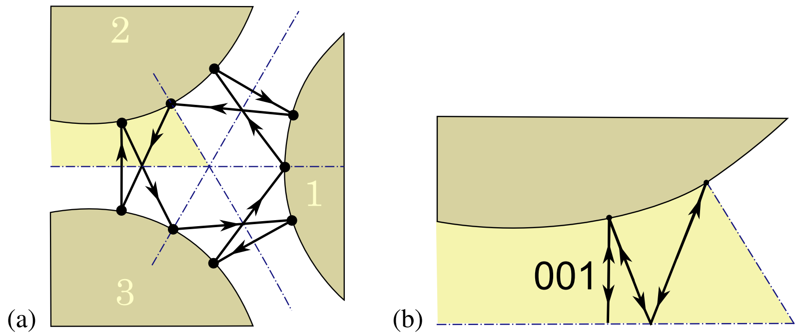


Figure 6.1: (a) The pair of full-space 9-cycles, the counter-clockwise $\overline{121232313}$ and the clockwise $\overline{131323212}$ correspond to (b) one fundamental domain 3-cycle $\overline{001}$.

Definition: Invariant subspace. $\mathcal{M}_\alpha \subset \mathcal{M}$ is an *invariant* subspace if

$$\{\mathcal{M}_\alpha \mid gx \in \mathcal{M}_\alpha \text{ for all } g \in G \text{ and } x \in \mathcal{M}_\alpha\}\,. \tag{6.3}$$

$\{0\}$ and $\mathcal{M}$ are always invariant subspaces. So is any $\mathrm{Fix}\,(H)$ which is point-wise invariant under action of G.

Definition: Irreducible subspace. A space $\mathcal{M}_\alpha$ whose only invariant subspaces under the action of G are $\{0\}$ and $\mathcal{M}_\alpha$ is called *irreducible*.

Example 6.1. ***Equivariance of the Lorenz flow.*** *The velocity field in Lorenz equations [2]*

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{6.4}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$C^{1/2}(x,y,z) = (-x,-y,z)\,. \tag{6.5}$$

Example 6.2. ***Desymmetrization of Lorenz flow:*** *(continuation of example 6.1) Lorenz equation (6.4) is equivariant under (6.5), the action of order-2 group $C_2 = \{e, C^{1/2}\}$, where $C^{1/2}$ is $[x,y]$-plane, half-cycle rotation by π about the z-axis:*

$$(x,y,z) \to C^{1/2}(x,y,z) = (-x,-y,z)\,. \tag{6.6}$$

$(C^{1/2})^2 = 1$ condition decomposes the state space into two linearly irreducible subspaces $\mathcal{M} = \mathcal{M}^+ \oplus \mathcal{M}^-$, the z-axis $\mathcal{M}^+$ and the $[x,y]$ plane $\mathcal{M}^-$, with projection operators onto the two subspaces given by

$$\mathbf{P}^+ = \frac{1}{2}(1 + C^{1/2}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{P}^- = \frac{1}{2}(1 - C^{1/2}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.7}$$

As the flow is C_2-invariant, so is its linearization $\dot{x} = Ax$. Evaluated at E_0, A commutes with $C^{1/2}$, and the E_0 stability matrix A decomposes into $[x,y]$ and z blocks.

Table 6.1: D_3 correspondence between the binary labeled fundamental domain prime cycles $\tilde{p}$ and the full 3-disk ternary labeled cycles p, together with the D_3 transformation that maps the end point of the $\tilde{p}$ cycle into the irreducible segment of the p cycle. White spaces in the above ternary sequences mark repeats of the irreducible segment; for example, the full space 12-cycle 1212 3131 2323 consists of 1212 and its symmetry related segments 3131, 2323. The multiplicity of p cycle is $m_p = 6n_{\tilde{p}}/n_p$. The shortest pair of fundamental domain cycles related by time reversal (but no spatial symmetry) are the 6-cycles $\overline{001011}$ and $\overline{001101}$.

$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$	$\tilde{p}$	p	$\mathbf{g}_{\tilde{p}}$
0	1 2	σ_{12}	000001	121212 131313	σ_{23}
1	1 2 3	C	000011	121212 313131 232323	C^2
01	12 13	σ_{23}	000101	121213	e
001	121 232 313	C	000111	121213 212123	σ_{12}
011	121 323	σ_{13}	001011	121232 131323	σ_{23}
0001	1212 1313	σ_{23}	001101	121231 323213	σ_{13}
0011	1212 3131 2323	C^2	001111	121231 232312 313123	C
0111	1213 2123	σ_{12}	010111	121312 313231 232123	C^2
00001	12121 23232 31313	C	011111	121321 323123	σ_{13}
00011	12121 32323	σ_{13}	0000001	1212121 2323232 3131313	C
00101	12123 21213	σ_{12}	0000011	1212121 3232323	σ_{13}
00111	12123	e	0000101	1212123 2121213	σ_{12}
01011	12131 23212 31323	C	0000111	1212123	e
01111	12132 13123	σ_{23}	$\cdots$	$\cdots$	$\cdots$

The 1-dimensional $\mathcal{M}^+$ subspace is the fixed-point subspace, with the z-axis points left point-wise invariant *under the group action*

$$\mathcal{M}^+ = \mathrm{Fix}\,(C_2) = \{x \in \mathcal{M} \mid g\,x = x \text{ for } g \in \{e, C^{1/2}\}\} \tag{6.8}$$

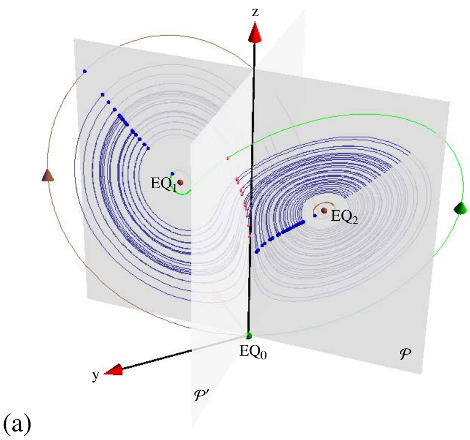
(here $x = (x, y, z)$ is a 3-dimensional vector, not the coordinate x). A C_2-fixed point $x(t)$ in $\mathrm{Fix}\,(C_2)$ moves with time, but according to (6.2) remains within $x(t) \in \mathrm{Fix}\,(C_2)$ for all times; the subspace $\mathcal{M}^+ = \mathrm{Fix}\,(C_2)$ is flow invariant. *In case at hand this jargon is a bit of an overkill: clearly for $(x, y, z) = (0, 0, z)$ the full state space Lorenz equation (6.4) is reduced to the exponential contraction to the E_0 equilibrium,*

$$\dot{z} = -b\,z\,. \tag{6.9}$$

However, for higher-dimensional flows the flow-invariant subspaces can be high-dimensional, with interesting dynamics of their own. Even in this simple case this subspace plays an important role as a topological obstruction: the orbits can neither enter it nor exit it, so the number of windings of a trajectory around it provides a natural, topological symbolic dynamics.

The $\mathcal{M}^-$ subspace is, however, not *flow-invariant, as the nonlinear terms $\dot{z} = xy - bz$ in the Lorenz equation (6.4) send all initial conditions within $\mathcal{M}^- = (x(0), y(0), 0)$ into the full, $z(t) \neq 0$ state space $\mathcal{M}/\mathcal{M}^+$.*

By taking as a Poincaré section any $C^{1/2}$-equivariant, non-self-intersecting surface that contains the z axis, the state space is divided into a half-space fundamental domain $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ and its 180^o rotation $C^{1/2}\tilde{\mathcal{M}}$. An example is afforded by the $\mathcal{P}$ plane section of the Lorenz flow in figure 6.1. Take the fundamental domain $\tilde{\mathcal{M}}$ to be the half-space between the viewer and $\mathcal{P}$. Then the full Lorenz flow is captured by re-injecting back into $\tilde{\mathcal{M}}$ any trajectory that exits it, by a rotation of π around the z axis.



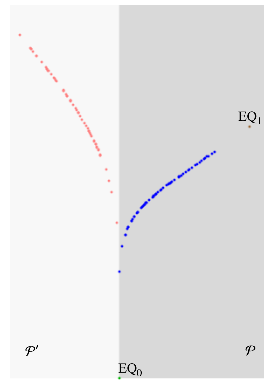


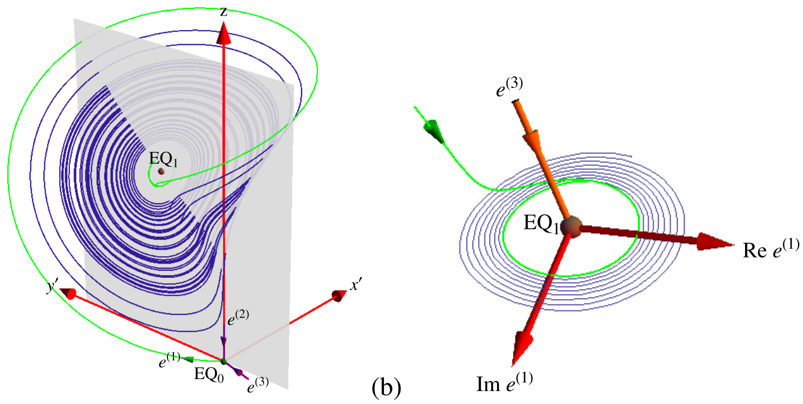
Figure 6.2: (a) Lorenz flow cut by $y = x$ Poincaré section plane $\mathcal{P}$ through the z axis and both $\mathrm{E}_{1,2}$ equilibria. Points where flow pierces into section are marked by dots. To aid visualization of the flow near the E_0 equilibrium, the flow is cut by the second Poincaré section, $\mathcal{P}'$, through $y = -x$ and the z axis. (b) Poincaré sections $\mathcal{P}$ and $\mathcal{P}'$ laid side-by-side. (E. Siminos)

As any such $C^{1/2}$-invariant section does the job, a choice of a 'fundamental domain' is here largely mater of taste. For purposes of visualization it is convenient to make the double-cover nature of the full state space by $\tilde{\mathcal{M}}$ explicit, through any state space redefinition that maps a pair of points related by symmetry into a single point. In case at hand, this can be easily accomplished by expressing (x, y) in polar coordinates $(x, y) = (r\cos\theta, r\sin\theta)$, and then plotting the flow in the 'doubled-polar angle representation:'

$$(\hat{x}, \hat{y}, z) = (r\cos 2\theta, r\sin 2\theta, z) = ((x^2 - y^2)/r, 2xy/r, z)\,, \tag{6.10}$$

as in figure 6.1 (a). In contrast to the original G-equivariant coordinates $[x, y, z]$, the Lorenz flow expressed in the new coordinates $[\hat{x}, \hat{y}, z]$ is G-invariant. *In this representation the $\tilde{\mathcal{M}} = \mathcal{M}/C_2$ fundamental domain flow is a smooth, continuous flow, with (any choice of) the fundamental domain stretched out to seamlessly cover the entire $[\hat{x}, \hat{y}]$ plane.*

(E. Siminos and J. Halcrow)



(a) (b)

Figure 6.3: (a) Lorenz attractor plotted in $[\hat{x}, \hat{y}, z]$, the doubled-polar angle coordinates (6.10), with points related by π-rotation in the $[x, y]$ plane identified. Stable eigenvectors of E_0: $\mathbf{e}^{(3)}$ and $\mathbf{e}^{(2)}$, along the z axis (6.9). Unstable manifold orbit $W^u(\mathrm{E}_0)$ (green) is a continuation of the unstable $\mathbf{e}^{(1)}$ of E_0. (b) Blow-up of the region near E_1: The unstable eigenplane of E_1 defined by $\mathrm{Re}\,\mathbf{e}^{(2)}$ and $\mathrm{Im}\,\mathbf{e}^{(2)}$, the stable eigenvector $\mathbf{e}^{(3)}$. The descent of the E_0 unstable manifold (green) defines the innermost edge of the strange attractor. As it is clear from (a), it also defines its outermost edge. (E. Siminos)

References

[1] S. M. Heilman and R. S. Strichartz, "Homotopies of eigenfunctions and the spectrum of the Laplacian on the Sierpinski carpet", Fractals **18**, 1–34 (2010).

[2] E. N. Lorenz, "Deterministic nonperiodic flow", J. Atmos. Sci. **20**, 130–141 (1963).

[3] M. Teytel, "How rare are multiple eigenvalues?", Commun. Pure Appl. Math. **52**, 917–934 (1999).

Exercises

6.1. **3-disk symbolic dynamics.** As periodic trajectories will turn out to be our main tool to breach deep into the realm of chaos, it pays to start familiarizing oneself with them now by sketching and counting the few shortest prime cycles. Show that the 3-disk pinball has $3 \cdot 2^{n-1}$ itineraries of length n. List periodic orbits of lengths 2, 3, 4, 5, $\cdots$. Verify that the shortest 3-disk prime cycles are 12, 13, 23, 123, 132, 1213, 1232, 1323, 12123, $\cdots$. Try to sketch them. (continued in exercise 6.3)

A comment about exercise 6.1, exercise 6.2, and exercise 6.3: If parts of these problems seem trivial - they are. The intention is just to check that you understand what these symbolic dynamics codings are - the main message is that the really smart coding (fundamental domain) is 1-to-1 given by the group theory operations that map a point in the fundamental domain to where it is in the full state space. This observation you might not find deep, but it is - instead of *absolute* labels, in presence of symmetries one only needs to keep track of *relative* motions, from domain to domain, does not matter which domain in absolute coordinates - that is what group actions do. And thus the word '*relative*' creeps into this exposition.

6.2. **Reduction of 3-disk symbolic dynamics to binary.** (continued from exercise 6.1)

(a) Verify that the 3-disk cycles
$\{\overline{1\,2}, \overline{1\,3}, \overline{2\,3}\}$, $\{\overline{1\,2\,3}, \overline{1\,3\,2}\}$, $\{\overline{12\,13}$ + 2 perms.$\}$,
$\{\overline{121\,232\,313}$ + 5 perms.$\}$, $\{\overline{121\,323}$+ 2 perms.$\}$, $\cdots$,
correspond to the fundamental domain cycles $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$ respectively.

(b) Check the reduction for short cycles in table 6.1 by drawing them both in the full 3-disk system and in the fundamental domain, as in figure 6.1.

(c) Optional: Can you see how the group elements listed in table 6.1 relate irreducible segments to the fundamental domain periodic orbits?

(continued in exercise 6.3)

6.3. **3-disk fundamental domain cycles.** Try to sketch $\overline{0}$, $\overline{1}$, $\overline{01}$, $\overline{001}$, $\overline{011}$, $\cdots$. in the fundamental domain, and interpret the symbols $\{0, 1\}$ by relating them to topologically distinct types of collisions. Compare with table 6.1. Then try to sketch the location of periodic points in the Poincaré section of the billiard flow. The point of this exercise is that while in the configuration space longer cycles look like a hopeless jumble, in the Poincaré section they are clearly and logically ordered. The Poincaré section is always to be preferred to projections of a flow onto the configuration space coordinates, or any other subset of state space coordinates which does not respect the topological organization of the flow.

6.4. **C_2-equivariance of Lorenz system.** Verify that the vector field in Lorenz equations (6.4)

$$\dot{x} = v(x) = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \sigma(y-x) \\ \rho x - y - xz \\ xy - bz \end{bmatrix} \tag{6.11}$$

is equivariant under the action of cyclic group $C_2 = \{e, C^{1/2}\}$ acting on $\mathbb{R}^3$ by a π rotation about the z axis,

$$C^{1/2}(x, y, z) = (-x, -y, z)\,,$$

as claimed in example 6.1.

6.5. **Proto-Lorenz system.** Here we quotient out the C_2 symmetry by constructing an explicit "intensity" representation of the desymmetrized Lorenz flow.

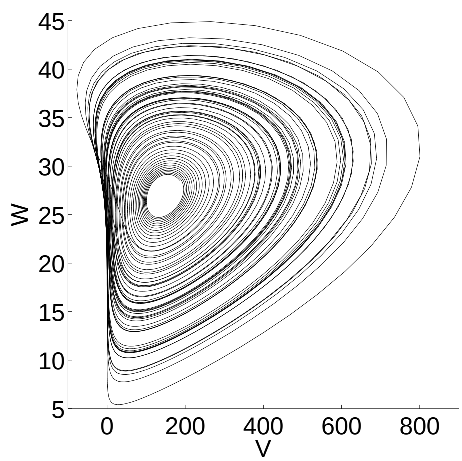
1. Rewrite the Lorenz equation (6.4) in terms of variables

$$(u, v, z) = (x^2 - y^2, 2xy, z)\,, \tag{6.12}$$

show that it takes form

$$\begin{aligned} \begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} -(\sigma+1)u + (\sigma-r)v + (1-\sigma)N + vz \\ (r-\sigma)u - (\sigma+1)v + (r+\sigma)N - uz - Nz \\ v/2 - bz \end{bmatrix} \\ N &= \sqrt{u^2+v^2}\,. \end{aligned} \tag{6.13}$$

2. Show that this is the (Lorenz)/C_2 quotient map for the Lorenz flow, i.e., that it identifies points related by the π rotation (6.6).
3. Show that (6.12) is invertible. Where does the inverse not exist?
4. Compute the equilibria of proto-Lorenz and their stabilities. Compare with the equilibria of the Lorenz flow.
5. Plot the strange attractor both in the original form (6.4) and in the proto-Lorenz form (6.13)



for the Lorenz parameter values $\sigma = 10$, $b = 8/3$, $\rho = 28$. Topologically, does it resemble more the Lorenz, or the Rössler attractor, or neither? (plot by J. Halcrow)

7. Show that a periodic orbit of the proto-Lorenz is either a periodic orbit or a relative periodic orbit of the Lorenz flow.
8. Show that if a periodic orbit of the proto-Lorenz is also periodic orbit of the Lorenz flow, their Floquet multipliers are the same. How do the Floquet multipliers of relative periodic orbits of the Lorenz flow relate to the Floquet multipliers of the proto-Lorenz?

9 Show that the coordinate change (6.12) is the same as rewriting

$$\begin{aligned} \dot{r} &= \frac{r}{2}(-\sigma - 1 + (\sigma + \rho - z)\sin 2\theta \\ &\quad +(1-\sigma)\cos 2\theta) \\ \dot{\theta} &= \frac{1}{2}(-\sigma + \rho - z + (\sigma - 1)\sin 2\theta \\ &\quad +(\sigma + \rho - z)\cos 2\theta) \\ \dot{z} &= -bz + \frac{r^2}{2}\sin 2\theta\,. \end{aligned} \tag{6.14}$$

in variables

$$(u, v) = (r^2 \cos 2\theta, r^2 \sin 2\theta)\,,$$

i.e., squaring a complex number $z = x + iy$, $z^2 = u + iv$.