group theory - week 2

Finite groups - definitions

Georgia Tech PHYS-7143

Homework HW2 due Thursday, January 28, 2016

== show all your work for maximum credit,
== put labels, title, legends on any graphs
== acknowledge study group member, if collective effort

Exercise 2.1 $G_x \subset G$	1 point
Exercise 2.2 *Transitivity of conjugation*	1 point
Exercise 2.3 *Isotropy subgroup of* gx	1 points
Exercise 2.5 C_4*-invariant potential*	7 (+2) points

Total of 10 points = 100 % score.

Bonus points

Exercise 2.8 *Three masses on a loop*	6 points
Exercise 2.7 *An arrangement of five particles*	4 points

Extra points accumulate, can help you later if you miss a few problems.

2016-01-19 Boris Lecture 3 Don't know group theory

Today's blackboard derivation of normal-modes of the ring of N asymmetric pairs of oscillators is taken from Gutkin lecture notes example 5.1 C_n *symmetry*. The corresponding projection operators (1.29) are worked out here, in example 2.4.

2016-01-21 Predrag Lecture 4 Finite groups

Groups, permutations, rearrangement theorem, subgroups, cosets, all exemplified by the $S_3 = C_{3v} = D_3$ symmetries of an equilateral triangle. This lecture follows closely Chapter 1 *Basic Mathematical Background: Introduction* of Dresselhaus *et al.* textbook [1] (click here, ask for password if you have forgotten it). This book (or Tinkham [3]) is good on discrete and space groups, but perhaps not so good on continuous groups. The MIT course 6.734 online version contains much of the same material.

If instead, bedside crocheting is your thing, click here.

2.1 Using group theory without knowing any

It's a matter of no small pride for a card-carrying dirt physics theorist to claim full and total ignorance of group theory (read sect. A.6 *Gruppenpest* of ref. [2]). So what we will do first is work out a few examples of physical applications of group theory that you already know without knowing that you have been using "Group Theory."

Example 2.1. ***Reflection and discrete rotation symmetries:***

(a) Reflection symmetry $V(x) = PV(x) = V(-x)$:

$$\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) = E_n\psi(x). \tag{2.1}$$

If $\psi(x)$ is solution then $P\psi(x)$ is also solution. From this and non-degeneracy of the spectrum follows that either $P\psi(x) = \psi(x)$ or $P\psi(x) = -\psi(x)$. The first case corresponds to symmetric functions while the second one to antisymmetric one. Thus the whole spectrum can be decomposed in accordance to symmetry group.

(b) Rotation symmetry $V(x) = gV(x)$, $G = \{e, g, g^2\}$: By the same argument we have three possibilities:

$$g\psi(x) = \psi(x); \qquad g\psi(x) = e^{i2\pi/3}\psi(x); \qquad g\psi(x) = e^{-i2\pi/3}\psi(x).$$

In addition, by the time reversal symmetry if $\psi(x)$ is solution then $\psi^*(x)$ is solution with the same eigenvalue as well. From this follows that the spectrum must be degenerate. The spectrum can be split into real eigenfunctions $\{\psi_1(x)\}$ invariant under rotations and degenerate pairs of real eigenfunctions:

$$\psi_2(x) = \psi(x) + \psi^*(x); \psi_3(x) = i(\psi(x) - \psi^*(x)), \text{ where } g\psi(x) = e^{i2\pi/3}\psi(x).$$

(B. Gutkin)

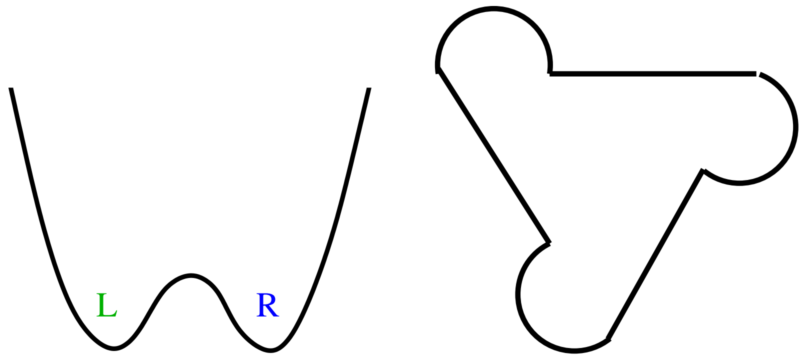


Figure 2.1: (left) A reflection-symmetric double-well potential. (right) A 3rd-circle rotation-symmetric plane billiard.

Example 2.2. ***Discrete symmetries in physics:***

- *Point groups i.e., subgroups of* $O(3)$.
- *Point groups + discrete translations e.g., symmetry groups of crystals.*
- *Permutation groups*

$$S\Psi(x_1, x_2, \ldots x_n) = \Psi(x_2, x_1, \ldots x_n).$$

- *Boson wave functions are symmetric while fermion wave functions are anti-symmetric under exchange of variables.*

(B. Gutkin)

(An extract from Gutkin lecture notes): Point groups are finite subgroups of $\mathrm{O}(3)$. We consider first finite subgroups G of $\mathrm{SO}(3)$ and then add special transformations.

1) Let H_A be a subgroup of a discrete group G which leaves invariant a point (pole) A on the unit sphere. $H_A \cong Z_{n_A}$, where n_A is the order of H_A. Consider the decomposition of G, $|G| = N$ into cosets:

$$G = \{g_1 \cdot H_A, g_n \cdot H_A, \ldots g_{m_A} \cdot H_A\}, \qquad m_A n_A = N\,. \tag{2.2}$$

each coset $g_i \cdot H_A$ transfers the point $A = A_1$ to some another point $A_i \neq A$ which is the same for all elements of the coset. Different cosets generate different points. The set $\{A_1, A_2, \ldots A_{m_A}\}$ is called **star** of A. If we start from another point A_i of the star we produce the same star. The transformations leaving A_i invariant have the form $g_i H_A g_i^{-1}$, and together they form a subgroup of G isomorphic to H_A.

Example 2.3. ***Star system of a tetrahedron:*** *The point group T of a tetrahedron has 4 axes of symmetries of 3rd order and 3 axes of 2nd order, see figure 2.2 (b). The corresponding star system is therefore:*

$$A = (A_1, A_2, A_3, A_4),\ [n_A = 3, m_A = 4]$$
$$B = (B_1, B_2, B_3, B_4),\ [n_B = 3, m_B = 4]$$
$$C = (C_1, C_2, C_3, C_4, C_5, C_6),\ [n_C = 2, m_C = 6]$$

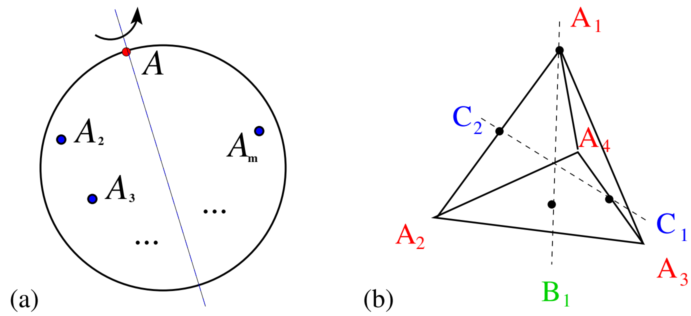


Figure 2.2: (a) Star of a point A. (b) Star system of the tetrahedron group.

Example 2.4. ***Projection operators for cyclic group C_N.***

Consider a cyclic group $C_N = \{e, g, g^2, \cdots g^{N-1}\}$, and let $M = D(g)$ be a $[2N \times 2N]$ representation of the one-step shift g. If $[V, M] = 0$, P_n are the same for the interaction potential V, and for the one-step shift matrix M. In the projection operator formulation (1.29), the N distinct eigenvalues of M, the Nth roots of unity $\lambda_n = \lambda^n$, $\lambda = \exp(i\,2\pi/N)$, split the $2N$-dimensional space into N invariant subspaces by means of projection operators

$$P_n = \prod_{m \neq n} \frac{M - \lambda_m I}{\lambda_n - \lambda_m} = \prod_{m=1}^{N-1} \frac{\lambda^{-n} M - \lambda^m I}{1 - \lambda^m} \,. \tag{2.3}$$

Using

$$(x - \lambda)(x - \lambda^2) \cdots (x - \lambda^{N-1}) = 1 + x + \cdots + x^{N-1} = \frac{1 - x^N}{1 - x} \,,$$

we obtain the projection operator in form of a discrete Fourier sum (rather than the product (1.29)),

$$P_n = \frac{1}{N} \sum_{m=0}^{N-1} e^{i \frac{2\pi}{N} mn} M^m \,.$$

As we'll teach you next, this is the simplest example of the general group theory machinery,

$$P_n = \frac{1}{|G|} \sum_{g \in G} \bar{\chi}(g) D(g) \,.$$

(B. Gutkin)

References

[1] M. S. Dresselhaus, G. Dresselhaus, and A. Jorio, *Group Theory: Application to the Physics of Condensed Matter* (Springer, New York, 2007).

[2] R. Mainieri and P. Cvitanović, in *Chaos: Classical and Quantum* (Niels Bohr Inst., Copenhagen, 2016) Chap. A brief history of chaos.

[3] M. Tinkham, *Group Theory and Quantum Mechanics* (Dover, New York, 2003).

Exercises

2.1. $G_x \subset G$. The maximal set of group actions which maps a state space point x into itself,

$$G_x = \{g \in G : gx = x\}\,, \tag{2.4}$$

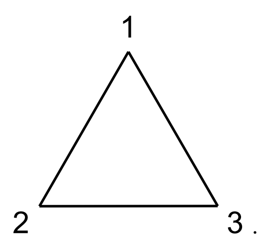
is called the *isotropy group* (or *stability subgroup* or *little group*) of x. Prove that the set G_x as defined in (2.4) is a subgroup of G.

2.2. **Transitivity of conjugation.** Assume that g_1, g_2, $g_3 \in G$ and both g_1 and g_2 are conjugate to g_3. Prove that g_1 is conjugate to g_2.

2.3. **Isotropy subgroup of gx.** Prove that for $g \in G$, x and gx have conjugate isotropy subgroups:

$$G_{gx} = g\,G_x\,g^{-1}$$

2.4. **D_3: symmetries of an equilateral triangle.** Consider group $D_3 \cong C_{3v}$, the symmetry group of an equilateral triangle:



(a) List the group elements and the corresponding geometric operations

(b) Find the subgroups of the group D_3.

(c) Find the classes of D_3 and the number of elements in them, guided by the geometric interpretation of group elements. Verify your answer using the definition of a class.

(d) List the conjugacy classes of subgroups of D_3. (continued as exercise 4.1)

2.5. **C_4-invariant potential.** Consider the Schrödinger equation for a particle moving in a two-dimensional bounding potential V, such that the spectrum is discrete. Assume that V is C_N-invariant, i.e., V remains invariant under the rotation R by the angle $2\pi/N$. It was explained in the lecture, that for $N = 3$ case, figure 2.3 (a), the spectrum of the system can be split into two sectors: $\{E_n^0\}$ non-degenerate levels corresponding to symmetric eigenfunctions $\phi_n(Rx) = \phi_n(x)$ and doubly degenerate levels $\{E_n^\pm\}$ corresponding to non-symmetric eigenfunctions $\phi_n(Rx) = e^{\pm 2\pi i/3}\phi_n(x)$.

Q 1 What is the spectral structure in the case of $N = 4$, figure 2.3 (b)?
How many sectors appear and what are their degeneracies?

Q 2 What is the spectral structure for general N?

Q 3 A constant magnetic field normal to the $2D$ plane is added to V.
How will it affect the spectral structure?

Q 4 (bonus question) Figure out the spectral structure if the symmetry group of potential is D_3 (also includes 3 reflections), figure 2.3 (c).

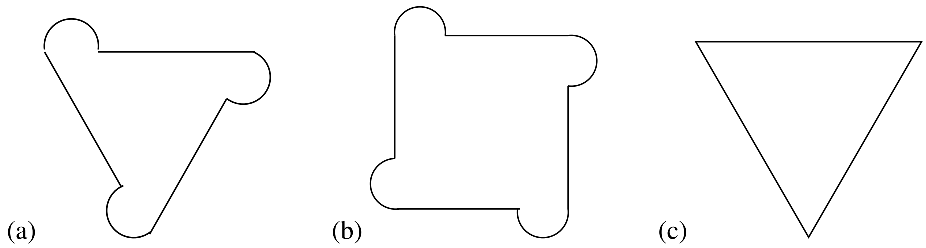


Figure 2.3: Hard wall potential with (a) symmetry C_3, (b) symmetry C_4, and (c) symmetry D_3.

(Boris Gutkin)

2.6. **Permutation of three objects.** Consider S_3, the group of permutations of 3 objects.

(a) Show that S_3 is a group.

(b) List the conjugacy classes of S_3?

(c) Give an interpretation of these classes if the group elements are substitution operations on a set of three objects.

(c) Give a geometrical interpretation in case of group elements being symmetry operations on equilateral triangle.

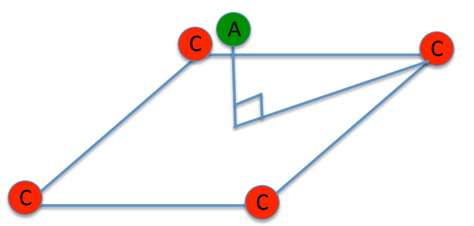


Figure 2.4: 4 identical particles of type C lie on the vertices of a square. In the center of the square, but out of the plane, is a particle of type A. (K. Y. Short)

2.7. **Arrangement of five particles.** Consider the arrangement of particles illustrated in figure 2.4: on each corner (vertex) of a rigid square lies a particle C; in the center of the square, but out of the plane on the z axis, is the particle A.

(a) What are the symmetries of this arrangement?

(b) Find its multiplication table.

(c) Find its subgroups.

(d) Determine the corresponding left and right cosets.

(e) Determine its conjugacy classes.

(f) Which subgroups are self-conjugate?

(g) Describe their factor groups.

(K. Y. Short)

2.8. **Three masses on a loop.** Three identical masses, connected by three identical springs, are constrained to move on a circle hoop as shown in figure 2.5. Find the normal modes. Hint: write down coupled harmonic oscillator equations, guess the form of oscillatory solutions. Then use basic matrix methods, i.e., find zeros of a characteristic determinant, find the eigenvectors, etc.. (K. Y. Short)

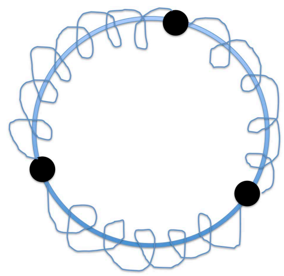


Figure 2.5: Three identical masses are constrained to move on a hoop, connected by three identical springs such that the system wraps completely around the hoop. Find the normal modes.